Vol. 89, No. 4, October 2016

MAA

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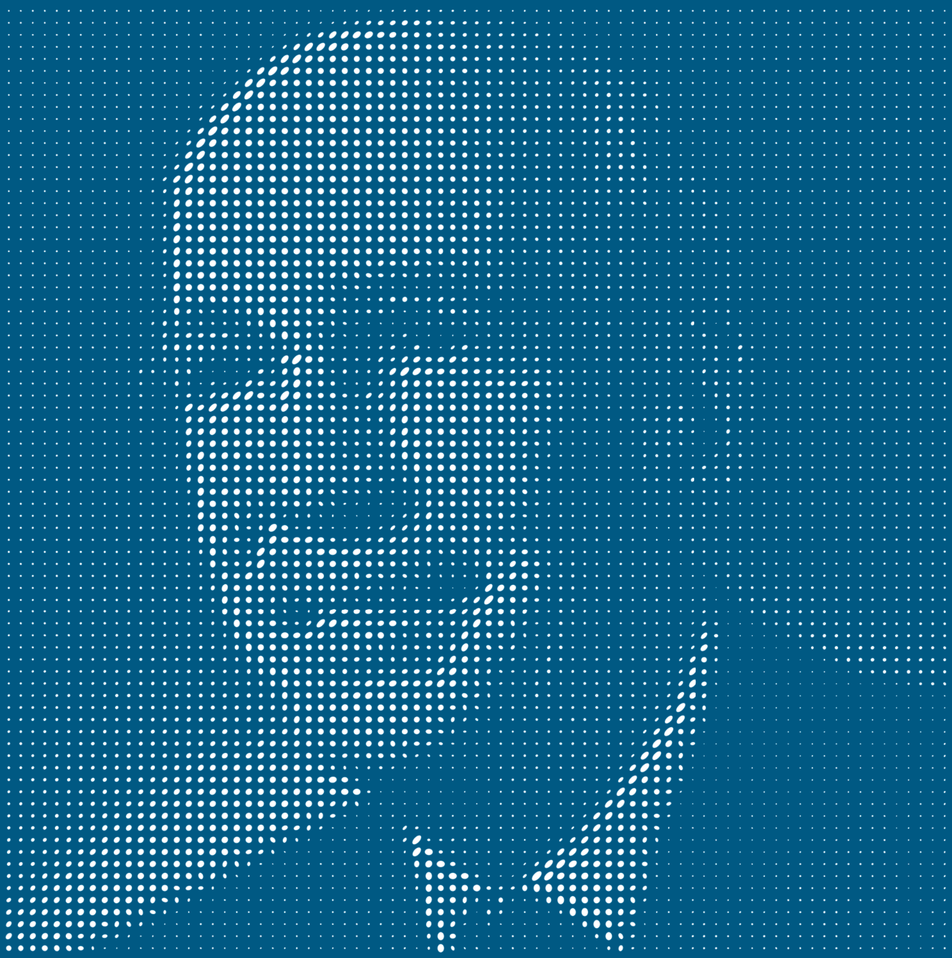

- Androids, ideal Nim, and Dickson's lemma
- Variations of Instant Insanity II
- Adding wild cards to poker
- Fuzzy logic on the island of knights and knaves

EDITORIAL POLICY

Mathematics Magazine aims to provide lively and appealing mathematical exposition. The *Magazine* is not a research journal, so the terse style appropriate for such a journal (lemma-theorem-proof-corollary) is not appropriate for the Magazine. Articles should include examples, applications, historical background, and illustrations, where appropriate. They should be attractive and accessible to undergraduates and would, ideally, be helpful in supplementing undergraduate courses or in stimulating student investigations. Manuscripts on history are especially welcome, as are those showing relationships among various branches of mathematics and between mathematics and other disciplines.

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MATHEMATICS MAGAZINE

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LETTER FROM THE EDITOR

Following the *College Mathematics Journal*'s annual September issue on recreational mathematics, the October MAGAZINE focuses on recreational mathematics in four articles. One of these is the article by Haley Dozier and John Perry, in which they introduce two generalizations of the combinatorial game Nim that are inspired by commutative algebra. Building on a *College Mathematics Journal* article by Richmond and Young from September 2013, Robert Beeler and Amanda Justus Bentley attempt to fix the Instant Insanity II puzzle. The "fix" is needed as Richmond and Young showed that there are two solutions, not one as claimed on the packaging. Beeler and Bentley analyze variations of the puzzle in an effort to yield a unique solution.

In Extreme Wild Poker, Ashley Fieldler, Carlie Maasz, Monta Meirose, and Christopher Spicer determine how many wild cards are necessary to add to a standard deck so that five-of-a-kind becomes the most common hand. Interspersed between the first three articles are two proofs without words. The next article by Jason Rosenhouse takes us to a mythical land where fuzzy knights and knaves live. These knights and knaves are the actors in Smullyan-like logic puzzles, but where fuzzy logic is used. After each of the first four articles are proofs without words. Two are by Ángel Plaza and one each by Jose Ángel Cid Araujo and Japheth Wood.

In the June 2016 letter from the editor, I incorrectly stated that Stephen Kaczkowski's article with three proofs of $\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^k = \frac{e}{e-1}$ was in the issue. For a number of reasons, we had to move the article to this issue. However, I failed to catch the change in the letter from the editor. I apologize for the confusion this caused. The article does appear in this issue where the three proofs involve calculus, real analysis, and Tannery's theorem.

Amy and Dave Reimann interview Dick Termes and ask him about the six-point perspective that he uses when he paints on a sphere. In advance of the US Presidential election, the crossword puzzle has a theme of the mathematics of elections. The latest problems and reviews are here, plus a correction for the April problems and corrections from one of the February 2016 Reviews. I, too, made a mistake in my description of the February Reviews; *mea culpa*.

The issue is completed by two news and letters items. The first announces the winners of the 2016 Carl B. Allendoerfer awards for the best expository writing in THIS MAGAZINE from 2015. Congratulations to Julia Barnes, Clinton Curry, Elizabeth Russell, and Lisbeth Schaubroeck for their article Emerging Julia Sets from April and to Irl Bivens and Ben Klein for their article The Median Value of a Continuous Function from February. Ellipses are used to represent Allendoerfer on the cover of the issue. Finally, the issue concludes with the problems, solutions, and results from both the 45th United States of America Mathematical Olympiad and 7th United States of America Junior Mathematical Olympiad, as well as a short memoriam to Jacek Fabrykowski, Professor of Mathematics, Youngstown State University, who was chair of the USAMO Committee from 2010 until his death on July 12, 2016. Details and results from the 57th International Mathematical Olympiad will appear in the December issue.

Michael A. Jones, Editor

ARTICLES

Androids Armed With Poisoned Chocolate Squares: Ideal Nim and Its Relatives

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One summer in the early 1980s, when "desktop" computers were a novelty that cost as much as a used car, and blocky, monochrome graphics were still state-of-the-art, the Hilton Public Library in Newport News, Virginia offered a prize to any young patron who could win a computer game named "Android Nim." The game featured three rows of robots. On each turn, the player chose a row and a number; the leftmost robot in the named row would fire a gun that disintegrated the named number of robots. On its turn, the computer likewise disintegrated a number of robots from one row. Whoever disintegrated the last robot won. The program knew the winning strategy, so even if you started at an advantage, one slip was enough to lose. Children could play unsupervised as long as they wanted, but had to win a supervised game to receive the prize. To our knowledge, no one won the prize, though one child did win an unsupervised game.

Android Nim is based on the game "Nim," for which Charles Bouton used simple mathematics to describe a winning strategy [**3**]. Following Bouton, mathematicians showed his ideas apply to an entire class of games called "impartial, perfect information games" which offer no element of chance (dice, spinner, ...); both players see all relevant game information (no hidden cards); and either player can select any available move (no distinction between "red" and "black" pieces).

This article introduces two impartial, perfect-information games inspired by ideas of commutative algebra. The first, *Ideal Nim*, generalizes *Chomp*, a known game which itself generalizes Nim. Ideal Nim's gameplay illustrates important algebraic ideas, allowing us to describe certain phenomena from a different point of view; we illustrate this on a famous result called Dickson's lemma. Adding a few rules to Ideal Nim leads to the second game, *Gröbner Nim*, in which players compute a Gröbner basis. Finding such a basis is a common task in most computer algebra systems, and is akin to asking the computer to "play" this game as "solitaire."

Nim and Chomp

In the game of Nim, two players (Emmy and David, say, with Emmy always going first) face several piles of pebbles. Players may remove any number of pebbles from

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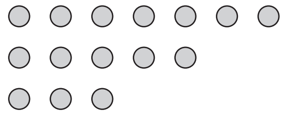
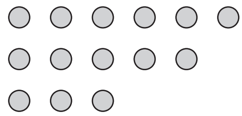
one pile, and the last player to take a pebble wins. Notice that if you replace the taking of pebbles by the disintegration of androids, you have Android Nim.

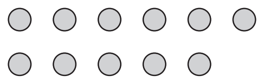
Example. Consider a game with piles of 7, 5, and 3 pebbles. It helps to visualize the piles as rows.



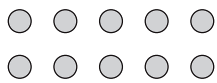
Suppose Emmy takes 1 pebble from the first row. That leaves 6, 5, and 3.



Suppose David takes all three pebbles in the last row. That leaves 6 and 5 pebbles.



Emmy now has an "obviously" good choice, to take one pebble from the first row.



Do you see why this choice leaves Emmy in an "obviously" good position? The resulting visual symmetry lets her mirror David's subsequent choice in one row with a balancing choice in the other. If David removes two from the first row, Emmy can restore symmetry by removing two from the second. Following this strategy, Emmy wins. ■

Can Emmy always guarantee a favorable outcome at the outset? Even if *visual* symmetry is impossible, she can usually impose *chronological* symmetry. Bouton advised the following strategy:

1. Express the number of pebbles in each row as a sum of distinct powers of two.
2. A game's nimber is the sum after canceling identical powers of two.
3. The best choice is to remove the number of pebbles necessary to reduce the nimber to zero.

(The term "nimber" is due to Conway [**2, 4**].)

Example (Continued). Applying this strategy to the initial configuration of the game above, we compute

$$\begin{aligned} 7 &= 1 + 2 + 4 \\ 5 &= 1 \qquad + 4 \\ 3 &= 1 + 2. \end{aligned}$$

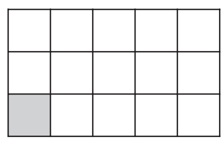
The best first move is to remove 1 pebble from any row, as that leaves us with powers of two that cancel, confirming Emmy's wisdom in removing a pebble in the first row. ■

It is well known that if the nimber is nonzero, Emmy can choose some combination of row and pebbles that reduces the nimber to zero. There is only one way to express a number of pebbles as a sum of distinct powers of two, so any subsequent move of David's necessarily "unbalances" the sum of powers, returning the game to a nonzero nimber value. Emmy can rebalance accordingly, reverting to the nimber of zero, which guarantees a win.

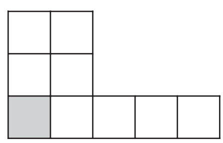
In the game of Chomp [**6**], two players agree to eat a block of chocolate divided into squares. The block can extend into any number of dimensions, but we typically keep ourselves to two, and locate squares using the natural lattice $\mathbb{N}^2 = \{(a, b) : a, b \in \mathbb{N}\}$. To orient ourselves, consider (a, b) northeast of (c, d) whenever $a \geq c$ (east) and $b \geq d$ (north); if it is, we similarly consider (c, d) southwest of (a, b). If the players like, the block of chocolate may extend infinitely north and east, and we can in fact view any finite block as an unfinished game on an infinite block.

Players break off squares according to the following rule: for any block they choose, they must consume that block, as well as every block "northeast" of it. We sometimes say these "gobbled" squares are "gone from gameplay," and lie in a set G, which is initially empty. In a grisly twist akin to Android Nim, the southwesternmost square at (0, 0) is poisoned; chomping it prompts a loss.

Example. Suppose David and Emmy play Chomp on a 3×5 block, whose poisoned square is shaded.



If Emmy chooses the third block in the middle row, she must also eat the third, fourth, and fifth blocks in the top and middle rows.



This is *not* a good move on Emmy's part, though we postpone an explanation until we have the tools for it. ■

Nim and Chomp are both impartial, perfect information games. The remarkable Sprague–Grundy theorem [**1, 2, 4, 7, 10**] states that all two-player, impartial, perfect information games are essentially variants on Nim, in that we can play them using the Bouton strategy of computing a nimber value and, if it is nonzero, play in such a way that the nimber value reduces immediately to zero.

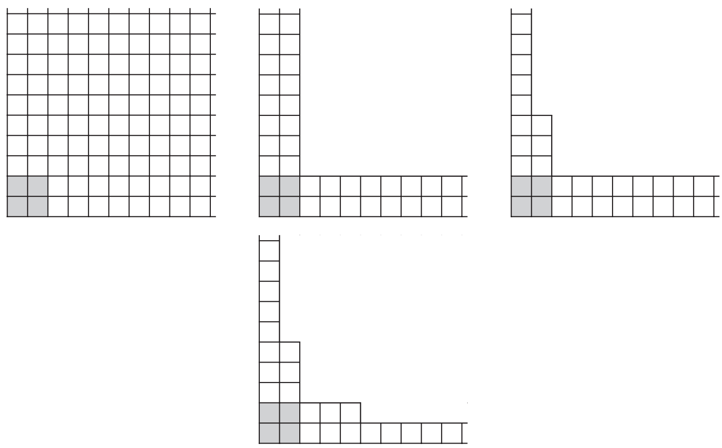
In theory, then, we can play Chomp by evaluating nimbers, and we will illustrate this on some easy examples later, but no one knows a practical way to compute nimbers for arbitrary configurations of Chomp [**8**]. Emmy and David might console themselves by playing Chomp in 3 or more dimensions; if they experiment just a little, they will find that any game of Nim on n rows can show up during a game of n-dimensional Chomp. So n-dimensional Chomp generalizes Nim.

Ideal Nim

Emmy and David have decided to experiment with Chomp a little further; they call the game "Ideal Nim" (we explain the name below).

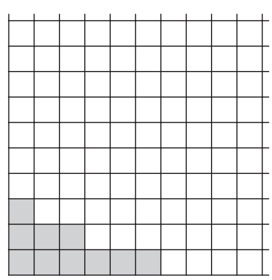
Rules of the game Rather than poison *just one* square in the lower left-hand corner, they first agree on a finite set F of lattice points, none of which lies northeast of another, and poison every square on the block that does not lie northeast of at least one point of F. Subsequent gameplay is identical to Chomp: Emmy and David select a nonpoisoned square to gobble blocks, and the last player to gobble a nonpoisoned square wins.

Example. Suppose Emmy and David select the finite set $F = \{(0, 2), (2, 0)\}$. This poisons the squares $(0, 0)$, $(0, 1)$ $(1, 0)$, $(1, 1)$. They then gobble the squares at $(2, 2)$, $(1, 5)$, and $(5, 1)$.

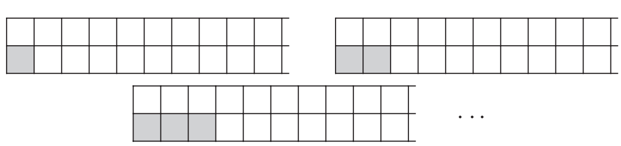


Emmy's first choice reduced the block to two healthy blocks that are independent in that gobbling a square in one block affects that block alone, and not the other. In a sense, her first move has created two visually symmetric games. This settles the game's outcome in her favor, as it did in the first game of Nim. Indeed, when David chose $(1, 5)$, Emmy replied with $(5, 1)$, restoring visual symmetry. ■

Again, Sprague–Grundy tells us Emmy can guarantee a favorable outcome by imposing a *chronological* symmetry, even when visual symmetry is unavailable. Consider the following configuration, which we call "Couch Potato."



To see how Emmy can win Couch Potato, consider the following sequence of seemingly unrelated games, which extend east indefinitely, though every square above the line $y = 2$ has been gobbled.



These games are unlike most of the others we consider in that David has the advantage, despite going second. Consider the following strategy, where $(n, 0)$ is the x-intercept in F:

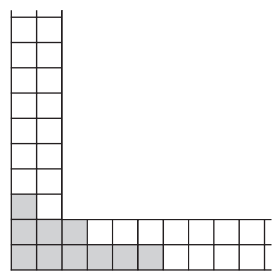
- If Emmy chooses $(m, 0)$, then David replies with $(m - n, 1)$.
- If Emmy chooses $(m, 1)$, then David replies with $(m + n, 0)$.

What happens? There are only two ways a player can win this game.

- Emmy can win at $(m, 0)$ only if $m = n$ and someone has already removed $(0, 1)$. But David chooses $(0, 1)$ only if Emmy first chooses $(n, 0)$, in which case the game ends after David's turn.
- Emmy can win at $(m, 1)$ only if $m = 0$ and someone has already removed $(n, 0)$. But David chooses $(n, 0)$ only if Emmy first chooses $(0, 1)$, in which case the game ends after David's turn.

Either way, Emmy loses.

On the bright side (for Emmy, anyway) this means Couch Potato *is* good news for Emmy. If she opens with $(2, 2)$, the game's new configuration is



Again, she has split the game into two healthy blocks that are independent—two games, really. They aren't visually symmetric, but they correspond to the first and third entries in the sequence above, which makes double trouble for David! Should he play in a row or column, Emmy can play in the same row or column using the strategy David used to win his games.

Example. Should David choose $(8, 0)$, Emmy restores balance by choosing $(5, 1)$; should David choose $(1, 7)$, Emmy restores balance by choosing $(0, 8)$. Eventually, she wins the game. ■

Since n-dimensional Ideal Nim generalizes n-dimensional Chomp, which generalizes n-row Nim, n-dimensional Ideal Nim also generalizes n-row Nim.

Ideal Nim as algebra Ideal Nim's inspiration comes from commutative algebra. This section outlines its relationship to objects called "monomial monoideals" [**9**], and concludes with a proof of Dickson's lemma based on David and Emmy's perspective.

A monoid is a pairing of a set M and an operation "$\star$" that satisfies the closure, associative, and identity properties on M. The operation need not be commutative in general, but we consider only commutative monoids.

Example. The set $\mathbb{N}$ of natural numbers is a "natural" example of a monoid in two different ways:

- under addition, with identity 0;
- under multiplication, with identity 1.

It is *not* a monoid under subtraction, as subtraction produces “unnatural” numbers ($2 - 3 \notin \mathbb{N}$).

Another example of a monoid is the natural lattice, under component-wise addition $(a, b) + (c, d) = (a + c, b + d)$; its identity is $(0, 0)$. ■

If M is a monoid under the operation $\star$, a subset I is a monoideal of M if the product of any $m \in M$ and any $i \in I$ lies in I; that is, $i \star m \in I$. Another way of saying this is that I absorbs multiplication by M.

Example. Let $a \in \mathbb{N}$. The set $A = \{n \in \mathbb{N} : n \geq a\}$ is a monoideal of $\mathbb{N}$ under addition, since $a + x \in A$ for any $x \in \mathbb{N}$.

Another example of a monoideal appears in Chomp and Ideal Nim. Each turn adds to G a set of squares that are gobbled away. The rules are defined in such a way that whenever a player selects (a, b), the points $(a + i, b + j)$ are subsequently out of play for every $i, j \in \mathbb{N}$. If we denote by $\langle G \rangle$ the set of points northeast of G, then $(a, b) + (i, j) \in \langle G \rangle$ for every $(a, b) \in \langle G \rangle$ and every $(i, j) \in \mathbb{N}^2$. In other words, $\langle G \rangle$ is a monoideal of $\mathbb{N}^2$. Hence the name “Ideal Nim.” ■

Another monoid is the set $\mathbb{T}$ of monomials in x and y under multiplication; the identity is $1 = x^0 y^0$. Monomial monoideals are of great interest in commutative algebra, and a monomial’s exponents are natural numbers, allowing us to identify every point of $\mathbb{T}$ with a unique point of $\mathbb{N}^2$. This matches the identity of $\mathbb{T}$ with that of $\mathbb{N}^2$, *and* products in $\mathbb{T}$ with the sum of corresponding points in $\mathbb{N}^2$. We call this relationship an isomorphism.

Example. This mapping sends $t = x^2 y^5$ to $a = (2, 5)$ and $u = y^7$ to $b = (0, 7)$. It likewise sends the product $tu = x^2 y^{12}$ to the sum $a + b = (2, 12)$. ■

The isomorphism between $\mathbb{T}$ and $\mathbb{N}^2$ means that $\langle G \rangle$ corresponds to a monoideal of $\mathbb{T}$. That is, Ideal Nim gives us a “recreational” perspective on monomial monoideals.

To illustrate this perspective, consider the following question. Emmy and David have some free time to play Ideal Nim, but they don’t want to spend an eternity on it. Do they run the risk of starting a game they might not finish, because it would drag on forever?

If you think about it, you can probably “see” that the answer is *no*; no matter how esoteric our initial set F, and no matter how Emmy and David select positions on the lattice, and no matter how many dimensions the lattice inhabits, every game will indeed end. To see why, consider the following well-known fact.

Theorem (Dickson’s Lemma, slightly restated). *Every subset of* $\mathbb{N}^n$ *has a finite subset of “maximally southwest” points.*

Dickson’s lemma implies that every game of Ideal Nim terminates. After all, the sequence S of points that Emmy and David actually choose during a game of Ideal Nim corresponds to a subset of $\mathbb{N}^n$. By Dickson’s lemma, that subset has a finite subset of maximally southwest points. *Emmy and David play all these points in finitely many turns!* Once they play the last point of that subset, every other point of S lies northeast of them, so the game must be over.

Is the converse true? That is, if we can show that every (n-dimensional) game of Ideal Nim terminates, does that also prove Dickson’s lemma? Indeed it does. Assume that we know every game of Ideal Nim terminates, and that, given any subset S of $\mathbb{N}^n$, Emmy and David agree to choose only points in S. As long as we can find points in S that are “more southwest” than points we’ve played previously, the game continues indefinitely. By hypothesis, however, the game cannot continue indefinitely, so Emmy and David run out of “maximally southwest” points in S after finitely many turns. We conclude that S has a finite subset of “maximally southwest” points.

We have just shown the following theorem.

Theorem. *Dickson's lemma is equivalent to termination of any game of Ideal Nim (in a finite number of dimensions).*

Can we prove that every game of Ideal Nim terminates, *without* reference to Dickson's lemma? Indeed we can. On each turn consider the set $\mathcal{P}$ of points that Emmy and David have played thus far. For each $k = 1, \ldots, n$, let $d_k = \min_{P \in \mathcal{P}} p_k$; that is, d_k is the smallest entry in the kth coordinate of a played point. The well ordering of $\mathbb{N}$ means that, when we recalculate d_k on subsequent turns, each d_k can decrease only finitely many times. Hence there exists a turn where all the d_k have reached their minimum values. At this point, let $D_k = \max_{P \in \mathcal{P}} p_k$; that is, D_k is the largest entry in the kth coordinate of a played point.

Suppose gameplay continues, which it can. A point Q is playable in a subsequent turn only if $q_k < D_k$ for at least one k. Only finitely many points Q (playable or not) satisfy the "squeezing" constraint that each $d_k \le q_k < D_k$, so a game that does not terminate must eventually select a playable point R such that $r_k \ge D_k$ for at least one k. Even this can occur at most finitely many times; to see why, let $K = \{k_1, \ldots, k_\ell\}$ be the set of "squeezed" indices k_i with $r_{k_i} < D_k$. Again, $d_{k_i} \le r_{k_i}$, and after playing R any playable point S with $r_{k_i} \le s_{k_i}$ is bounded above in its other, "nonsqueezed" coordinates $(\{1, \ldots, n\} \setminus K)$. Indeed, the bound is strict, as we must have at least one $s_\ell < r_\ell$. The set of playable points of this type, then, is also finite. To sum up, there are only finitely many points squeezed between the d_k and D_K, and only finitely many *playable* points not squeezed between them. This shows the following theorem.

Theorem. *Any game of Ideal Nim in finitely many dimensions must end.*

The discussion above means we have a game-based proof of Dickson's lemma.

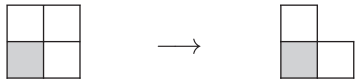
Ideal Nim as a game To win a game of Ideal Nim, Emmy and David need a reliable way to evaluate the nimber value of configurations of an arbitrary impartial, perfect information game. One way is to look at a player's options and decide the value of a configuration recursively. The base case is the configuration that offers no options, because the game is over. It makes sense to call its nimber value 0. For games with *some* options, we evaluate their value according to the mex rule ("minimum excludant"): if configuration C has options with nimber values indexed by $A = \{a_\lambda\}_{\lambda \in \Lambda}$, then the value of C is the smallest nimber c such that $c \notin A$. If A's minimum excludant is 0, we say C is a zero game.

Example. This definition is compatible with the values of a pile of pebbles in Nim.

- An empty pile (nimber 0) has no options, so $A = \{\}$. The smallest nimber that does not appear in A is 0.
- Inductively, then, a pile with n pebbles (nimber n) offers n possibilities: remove one, two, …, or n pebbles. This reduces the nimber value to zero, one, …, or $n - 1$, so $A = \{0, 1, \ldots, n - 1\}$. The smallest nimber that does not appear in A is n. ■

A pile of Nim objects corresponds to a row of points in $\mathbb{N}^2$, hence to an isolated row of squares in Chomp or Ideal Nim, so the example above resolves single rows of squares in both games. Chomp and Ideal Nim offer other shapes, and the mex rule applies to them, as well.

Example. A 2×2 block of chocolate with the lower left square poisoned has three options. The options $(0, 1)$ and $(1, 0)$ both remove two squares, leaving one healthy square, which has nimber value 1. The option $(1, 1)$ removes one square, leaving two healthy squares which are two, independent blocks:

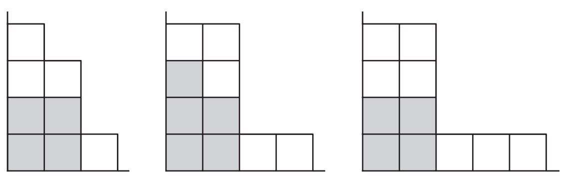


Whichever block David chooses from this new configuration, Emmy can choose the other to win, so the new configuration has nimber value 0. With options of value 0 and 1, the original configuration has value 2. Similar reasoning shows that these configurations of Ideal Nim have values 1 and 3, respectively:

Finally, if you look back at the 3×5 game of Chomp where Emmy made a bad first move, you can use the mex rule to confirm that Emmy's move results in a configuration whose nimber is 1. ■

The mex rule bogs down quickly in recursion. An alternative technique works by adapting the principle of symmetry we used in Nim and several examples of Ideal Nim. Chronological symmetry gives us an important property for the sum of two games. Suppose Emmy and David play two games with nimber values x and y. We define $x + y$ to be the nimber value of the game defined by playing both simultaneously. If $x = y$, then Emmy and David face two equivalent configurations simultaneously, and David can mimic any move Emmy makes in one configuration—just as with the two piles of 5 objects we saw in the first game! This game is chronologically symmetric—a zero game, so we define $x + x = 0$. On the other hand, if $x \neq y$, then the mex rule says we can reduce the larger of x or y to the smaller; that is, one option of $x + y$ is the zero game $x + x$ or $y + y$. So if $x \neq y$, the game is not yet chronologically symmetric, and cannot be a zero game: $x + y \neq 0$. So $x + y = 0$ if and only if $x = y$. We can use this fact to compute the values of harder games from smaller ones.

Example. We illustrate this on a few configurations of Ideal Nim.

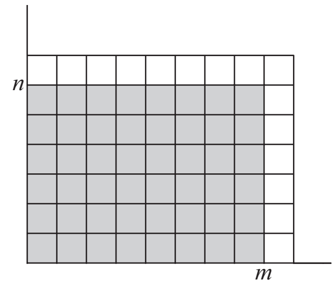


Each configuration above has two independent, playable regions; we'll call the upper region's nimber value u and the lower region's nimber value ℓ. From left to right, the values of ℓ are 1, 2, and 3. Each game is also a zero game; we leave the verification of this fact for the leftmost configurations to the reader, but for the rightmost configuration,

if Emmy chooses:	...then David should choose:	...which reduces the game to:
(0, 2)	(2, 0)	$0 + 0$
(0, 3)	(4, 0)	$2 + 2$
(1, 2)	(4, 0)	$2 + 2$
(1, 3)	(3, 0)	$1 + 1$

This confirms that $\ell + u = 0$. We explained earlier that $\ell = 3$; it follows that $u = 3$, as well. This agrees with the previous example's computations of the value of the upper region using the mex rule. ■

Both tools help with more complicated examples. The following example is a possible simplification of nearly all configurations.



(Positions (x, y) with $x > m$ or $y > n$ were already gobbled away.) Any valid move from this configuration removes from gameplay at least the point (m, n) in the northeast corner, so evaluating the configuration amounts to computing $x + y$ for all "row games" x and y such that $x \leq m$ and $y \leq n$. These "row games" are equivalent to Nim heaps. Between the mex rule and nimber arithmetic, we derive a simple rule for finding the value of the configuration. Without loss of generality, assume $m \geq n$.

1. Write m and n in binary as $m_\ell m_{\ell-1} \cdots m_0$ and $n_\ell n_{\ell-1} \cdots n_0$ ($m_\ell \neq 0$);
2. find the rightmost index i such that $m_i = n_i = 0$; in the worst case, $i = \ell + 1$;
3. moving left from i, find the leftmost index j such that $m_j = 1$ and $n_j = 0$, unless there is none, in which case let $j = i$;
4. the value of the game will be $\vartheta = 1 \cdot 2^j + a_{j-1} \cdot 2^{j-1} + \cdots + a_{i+1} \cdot 2^{i+1} + 1 \cdot 2^i$, where $a_k = 1$ if $m_k \neq n_k$, and $a_k = 0$ otherwise.

In the case $m = n$, this rule results in the smallest power of 2 that does not appear in their binary expansion. In the case $m = 2^k$ and $n = 2^k - 1$, it results in 2^{k+1}.

To see why the rule makes sense, apply the mex rule. We consider two separate cases.

- When $i = \ell + 1$, we can move in either the row of length m or the column of length n to obtain any value smaller than ϑ.
- Suppose $i \neq \ell + 1$. Whenever $k > j$, n_k must be identical to m_k; otherwise $m < n$, contradicting the choice of j.
  - So we can obtain any value *smaller* than ϑ by removing m_j and, if we need to change m_k for some $k < j$, adding or removing m_k as needed. (After all, the sum of all these m_k's is smaller than m_j.)
  - To obtain ϑ itself, one would need to move *only* in the m line or *only* in the n line to a position whose binary expansions change the value of m_i or n_i from 0 to 1. When $k > j$, we cannot borrow from m_k or n_k because the property $m_k = n_k$ would change, so the value of the game would be $2^k +$ smaller terms $> \vartheta$. To get ϑ, we also have to preserve the values of a_k for $k = j, j-1, \ldots, i+1$, so we cannot change m_i or n_i by borrowing from any of those terms. On the other hand, it is impossible to increase m_i (or n_i) by decreasing only those m_k (or n_k) where $k < i$. Thus ϑ is not an option of the current configuration, which makes it the minimal excluded value.

Gröbner Nim

Emmy and David now wonder if they can modify Ideal Nim, a game with connections to monomial monoideals, in a way that yields a game with connections to polynomial ideals. They call the resulting game "Gröbner Nim."

Rules of the game. Gröbner Nim begins with arbitrary "sticks" that connect some points of the natural lattice. For simplicity, the example sticks connect only two points, but we could use longer sticks. Gameplay depends on a stick's "distinguished point," whose coordinates have the largest sum, called the point's degree. If two or more points' degrees are the same, we distinguish the point that lies furthest right. When we refer to a stick AB, the first endpoint (A) is the distinguished point. This rule to distinguish a point creates a well ordering of $\mathbb{N}^2$, which will prove useful in a moment.

Example. A stick defined by the points $P = (3, 5)$ and $Q = (2, 3)$ has P as its distinguished point, so we would refer to it as PQ, never as QP. A stick defined by $R = (1, 2)$ and $S = (3, 0)$ has S as its distinguished point, so we would refer to it as SR, never as RS. ■

On each turn, a player produces a new stick. Each turn has at least one stage, *generation*, and usually a second, *reduction*.

- At *generation*, a player selects two sticks, say AB and CD, then shifts north or east a copy of AB, CD, or both to the smallest possible $A'B'$ and $C'D'$ such that $A' = C'$, and eliminates the meeting point $A' = C'$. Neither player may select both AB and CD again, although AB or CD may be selected with a different stick.
  - If $B' = D'$, the player discards them, and the turn ends.
  - Otherwise, the player connects them to form a new stick, and moves to the next stage.
- At *reduction*, the player checks whether B' or D' lies northeast of the distinguished point E of any pre-existing stick EF.
  - If not, the turn ends.
  - Otherwise, suppose B' lies northeast of E. The player shifts northeast a copy of EF to $E'F'$ such that E' meets B', and again eliminates the meeting point.
    - ∗ If the remaining points coincide, the player discards them, and the turn ends.
    - ∗ Otherwise, the player connects them to form a new stick and returns to the beginning of reduction with the new stick in place of $B'D'$.

Generating the last stick wins, even if the player discards it, so the first player unable to generate a new stick loses.

Example. Figure 1 shows both stages in the first turn of one possible game, beginning with a horizontal stick, AB, and a diagonal stick, CD.

At *generation*, Emmy has no choice but to shift copies of AB and CD to meet at the point (3, 2). She eliminates the distinguished points, resulting in a new stick that joins the points (2, 3) and (0, 2). We'll call this stick EF.

At *reduction*, Emmy observes that E lies northeast of C, so she shifts a copy of CD northeast to make EG, where $G = (1, 4)$. She eliminates the meeting point and connects GF. Since G still lies northeast of CD, she shifts a copy of CD north to make GH, where $H = (0, 5)$. She eliminates the meeting point and connects HF. Neither F nor H lies northeast of any other distinguished point, so Emmy's turn ends.

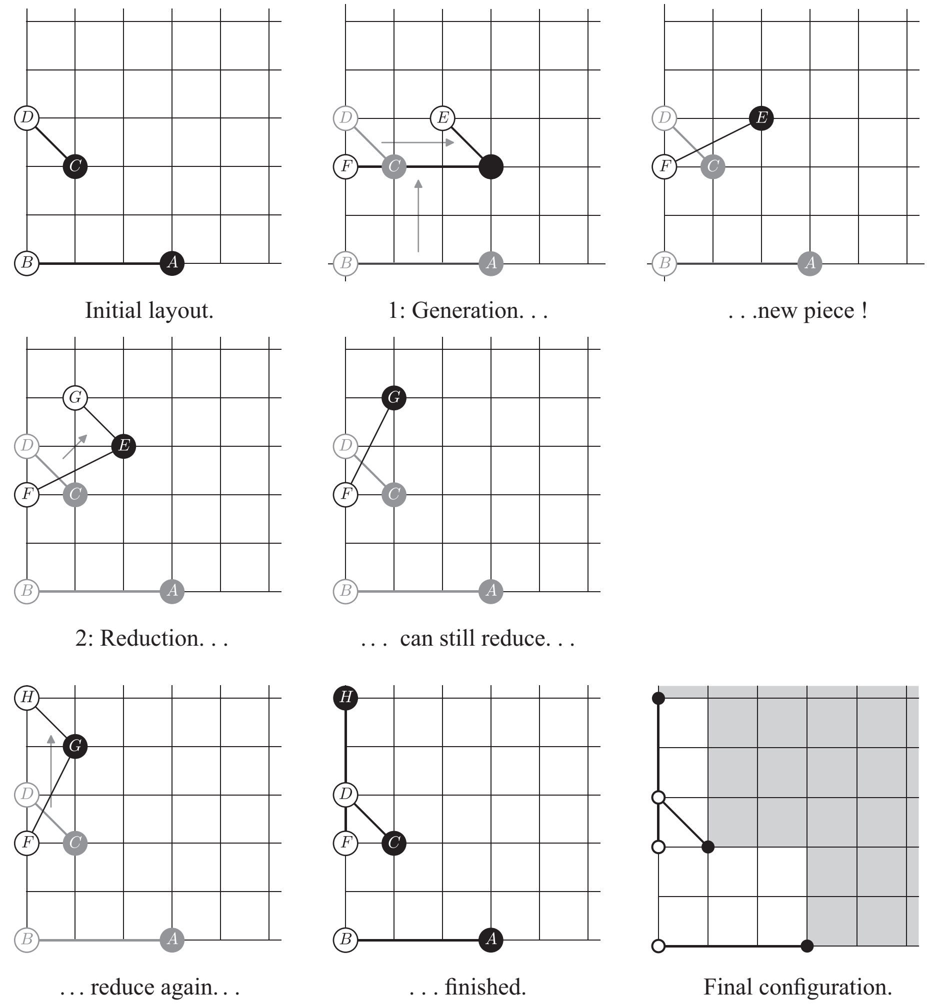


Figure 1 The first turn of a game of "Gröbner Nim." In stage 1, *generation*, the player creates a new stick. In stage 2, *reduction*, the same player eliminates points that lie northeast of an older stick.

The figure stops with the end of Emmy's turn, but let's consider David's options. He cannot choose AB and CD, because Emmy already played them; that leaves AB and HF, or CD and HF. Suppose he chooses AB and HF:

- At *generation*, David shifts a copy of AB north, and a copy of HF east, to meet at (3, 5). He eliminates the meeting point, and connects the remaining points to obtain a new stick IJ, where $I = (3, 2)$ and $J = (0, 5)$.
- At *reduction*.
  - Since I lies north of A, David shifts a copy of AB north to meet it. He eliminates the meeting point, and connects the remaining points to obtain a new stick JK, where $K = (0, 2)$.
  - Since J lies north of H (0 units north is still north by our rules), David shifts a copy of HF north to meet it. He eliminates the meeting point. In this case, the remaining points F and K coincide, so the stick collapses into nothingness. His turn ends.

- You may have noticed that, at the beginning of reduction, David also had the choice to shift a copy of CD east to meet A. In many cases this choice affects the game's configuration at the end of a turn, but we leave it to you to verify that this does not happen here.

It can be shown that, *regardless of how Emmy and David play*, the game ends with the configuration shown in the final diagram of Figure 1. ∎

Gröbner Nim as a game. This new game is clearly an impartial, perfect-information game. Unfortunately, reduction makes it a complicated one to analyze, indeed impractical in most cases. We highlight instead how Gröbner Nim is really a variant of Ideal Nim with some extra rules that restrict the choice of a nongobbled lattice point.

Focus on the elimination stage of each turn: if a point of the newly generated stick lies northeast of a pre-existing point's distinguished point, then the player must reduce it. That forces each new stick to have all its points not-northeast of a pre-existing point, just as players in Ideal Nim must choose a point not-northeast of a gobbled point. In other words, the distinguished points at the end of each term define a set G, that in turn defines a monoideal of gobbled points that are gone from gameplay! Figure 1 highlights this monoideal in its shaded region.

Any game of Gröbner Nim also features a predetermined "forbidden frontier" F; we see it as the *unshaded* region of Figure 1. While F's initial invisibility might seem to violate the "perfect information" requirement of these games, it is in fact defined unambiguously by the set of initial sticks, and the players *could* compute it at the beginning, if they so chose. As with a complete analysis of nimber values of a game's options, however, this is typically too burdensome to perform by hand.

Gröbner Nim as algebra. As you may have guessed, the reduction stage can take a long time. Will it proceed indefinitely? No, thanks to the ordering that determines a distinguished point: any new point is itself smaller than the eliminated point, either in degree or in its x-position. These are both natural numbers, so it isn't hard to see that we have a well ordering.

In addition, the use of sticks takes us beyond monoids and monoideals. A ring is a set we consider with two operations, addition and multiplication. The ring satisfies the requirements of a monoid under both operations, while addition must also satisfy the commutative and inverse properties. The two operations interact via the distributive property.

Example. The set $\mathbb{Z}_2[x, y]$ of polynomials in x and y with integer coefficients modulo 2 satisfies these properties. For instance, $x^2 + xy$ is its own additive inverse, and polynomial multiplication distributes over polynomial addition. ∎

Just as monoids have monoideals, rings have ring ideals, subsets that remain closed under subtraction and absorb products with any element from the ring.

Example. Let $I = \left\{p\left(x^3+1\right) + q\left(xy^2+y^3\right) : p, q \in \mathbb{Z}_2[x, y]\right\}$. We claim I is an ideal of $\mathbb{Z}_2[x, y]$:

- For any $f, g \in I$, choose $p, q, r, s \in \mathbb{Z}[x, y]$ so that $f = p\left(x^3+1\right) + q\left(xy^2+y^3\right)$ and $g = r\left(x^3+1\right) + s\left(xy^2+y^3\right)$. Properties of the ring and ideal show that $f - g = (p-r)\left(x^3+1\right) + (q-s)\left(xy^2+x^3\right) \in I$.
- For any $f \in I$ and any $g \in \mathbb{Z}_2[x, y]$, write f as above. Properties of the ring and ideal show that $gf = (gp)\left(x^3+1\right) + (gq)\left(xy^2+y^3\right) \in I$. ∎

You may notice a resemblance between the definition of I above and that of a vector space. A nice property of a finite-dimensional vector space V of dimension m over a field $\mathbb{F}$ is that we can write any vector $\mathbf{u}$ in terms of the basis elements $\mathbf{v}_1, \ldots, \mathbf{v}_m$ by finding scalars $a_1, \ldots, a_m \in \mathbb{F}$ such that

$$\mathbf{u} = a_1\mathbf{v}_1 + \cdots + a_m\mathbf{v}_m.$$

It would be nice if we could similarly find a finite "basis" of any ideal A, $\{a_1, \ldots, a_m\}$, which would allow us to express any element b of an ideal A in terms of the basis using ring elements $r_1, \ldots, r_m \in R$, as

$$b = r_1 a_1 + \cdots r_m a_m.$$

Alas, this is not always possible. Rings where every ideal satisfies the finite basis property are called Noetherian (for Emmy Noether), and the Hilbert basis theorem (for David Hilbert) states that if a ring R of coefficients is Noetherian, then so is any ring of polynomials (in *finitely* many variables) with coefficients from R.

Example. The ring $\mathbb{Z}_2 = \{0, 1\}$ under addition and multiplication (modulo 2) has exactly two ideals, $\{0\}$ and $\mathbb{Z}_2$. The basis for the former is $\{0\}$; the basis for the latter is $\{1\}$. Since every ideal of $\mathbb{Z}_2$ has a finite basis, the Hilbert basis theorem tells us that every ideal of $\mathbb{Z}_2[x, y]$ has a finite basis. ■

A common proof of the Hilbert basis theorem considers a polynomial's "leading monomial." For the sake of this discussion, a leading monomial has the largest sum of exponents; to break ties, use the largest degree in x_1; to break remaining ties, use the largest degree in x_2; etc. This extends our well ordering of $\mathbb{N}^2$ using "distinguished points" via our isomorphism between $\mathbb{T}$ and $\mathbb{N}^2$. Let R be a Noetherian ring, P a polynomial ring whose coefficients are in R, and I an ideal of P. The proof proceeds in three steps [**5, 9**].

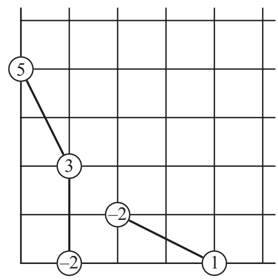
1. The monoideal T generated by the "leading monomials" of polynomials in I satisfies a Noetherian property: we can identify $t_1, \ldots, t_m \in T$ such that for any $u \in T$, we can find $i \in \{1, \ldots, m\}$ and a monomial v such that $u = vt_i$. (The reader should recognize this as Dickson's lemma, under the isomorphism between $\mathbb{T}$ and $\mathbb{N}^2$.)
2. Let $p_1, \ldots, p_m$ be polynomials of I whose leading monomials are $t_1, \ldots, t_m$. Using our well ordering of monomials, we can in finite time write any polynomial $f \in I$ in the form $q_1 p_1 + \cdots + q_m p_m + r$, where the q's and r come from P, *and* if $r \neq 0$, then its leading monomial is not in T.
3. In fact, closure of subtraction guarantees that $r = f - (q_1 p_1 + \cdots + q_m p_m) \in I$; since r cannot have a leading monomial in T, it must be that $r = 0$. So P is Noetherian.

Not all bases of vector spaces are convenient to analyze their structure, and a major topic of undergraduate linear algebra is the Gauss–Jordan algorithm to find a convenient basis. In the same way, not all finite bases of polynomial ideals are convenient to analyze their structure, and a major topic of commutative algebra is Buchberger's algorithm to find a Gröbner basis—hence the name, Gröbner Nim, as it presents a simplified version of Buchberger's algorithm.

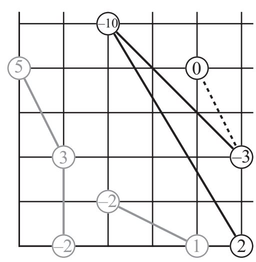
Example. Thanks to the isomorphism between $\mathbb{T}$ and $\mathbb{N}^2$, we can view the sticks of Figure 1 as the binomials $x^3 + 1$ and $x^2y + y^3$, which form a basis for the ideal I mentioned in a previous example. The sticks' "distinguished points" correspond to the leading monomials x^3 and x^2y. The game's rules guide us through Buchberger's algorithm in the self-canceling ring $\mathbb{Z}_2$, and the algorithm discovers a polynomial $y^5 + y^2$ in I, whose leading monomial is divisible by neither x^3 nor x^2y. This indivisibility is why the original basis is inconvenient, but once we include $y^5 + y^2$ in the basis, the leading terms of all other polynomials in I are divisible by x^3, x^2y, and y^5.

While we have couched the rules in terms of an ideal generated by bivariate polynomials over $\mathbb{Z}_2$, we could generalize to more variables or to "colored" points that correspond to coefficients in Noetherian rings other than $\mathbb{Z}_2$. Indeed, most computer algebra systems "play" Gröbner Nim "solitaire" when users ask them to compute a Gröbner basis. Experiment a little to see why you rarely want to play it with a friend.

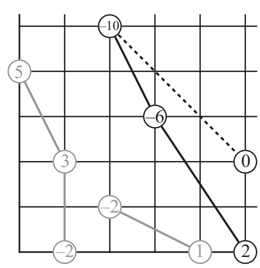
Example. Emmy and David decide to play the game with two sticks of colored points. One stick connects (4, 0) (color 1) and (2, 1) (color −2); the other connects (0, 5) (color 5), (1, 2) (color 3), and (0, 0) (color −2).



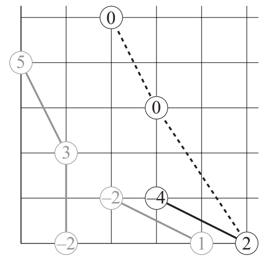
Emmy chooses the only pair available. When generating the new polynomial, she not only has to move the sticks north or east, she also has to adjust the colors so that the distinguished points cancel; she does this by multiplying by the number that makes the colors cancel when they are added together. She can do this by multiplying the colors of the shorter tree by 5, and that of the longer tree by −1.



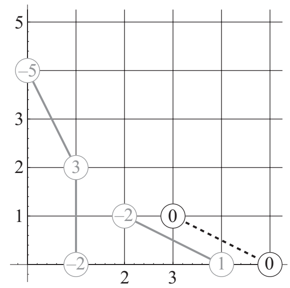
Emmy moves to reduction; the new stick's distinguished point lies northeast of the shortest stick's distinguished point, so she eliminates that.



The new stick's distinguished point lies northeast of the second stick's distinguished point, so she eliminates that, and a second point for free.



The new stick's distinguished point lies northeast of the shortest stick's distinguished point, so she eliminates that, and in fact she eliminates the stick altogether.



It is now David's turn, but the only trees on the board are the two that Emmy already chose. David cannot generate a new tree, so he loses. ■

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Summary. We describe two new combinatorial games. The first, Ideal Nim, both generalizes the well-known game Nim and its relative Chomp, and provides a recreational perspective on some important ideas of commutative algebra; for instance, the fact that the game is guaranteed to end is equivalent to Dickson's lemma, a well-known fact of commutative algebra. This relationship leads to a game-based proof of Dickson's lemma. The second game, Gröbner Nim, is really a variant of Ideal Nim that illustrates Buchberger's algorithm to compute a Gröbner basis. We conclude by describing the relationship between Gröbner Nim and polynomial rings.

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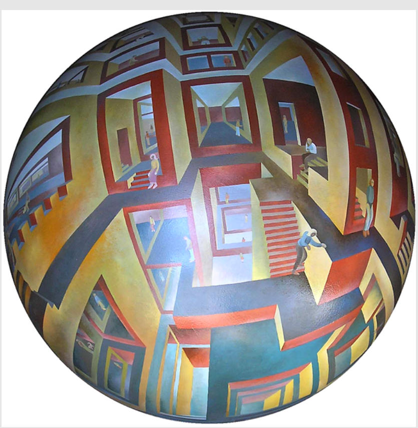
Artist Spotlight
Dick Termes

Which Way, Dick Termes; 12-inch painted sphere, 2003. Thirty diamond shapes make up this sphere. These diamonds turn into cubical rooms for people, plants, and animals to live in. Some of the cubes work very normal but the bottom and the top of the sphere become optical illusions where each side can be read with two different cubes. The people in these area are confused as to where they belong.

See interview on page 290.

Artist Spotlight
Dick Termes



Emptiness, Dick Termes; 24-inch painted sphere, 1986. This sphere shows rooms within rooms within rooms around you. Each room has one person which shows another type of emptiness.

See interview on page 290.

Proof Without Words: The Parallelogram With Maximum Perimeter for Given Diagonals Is the Rhombus

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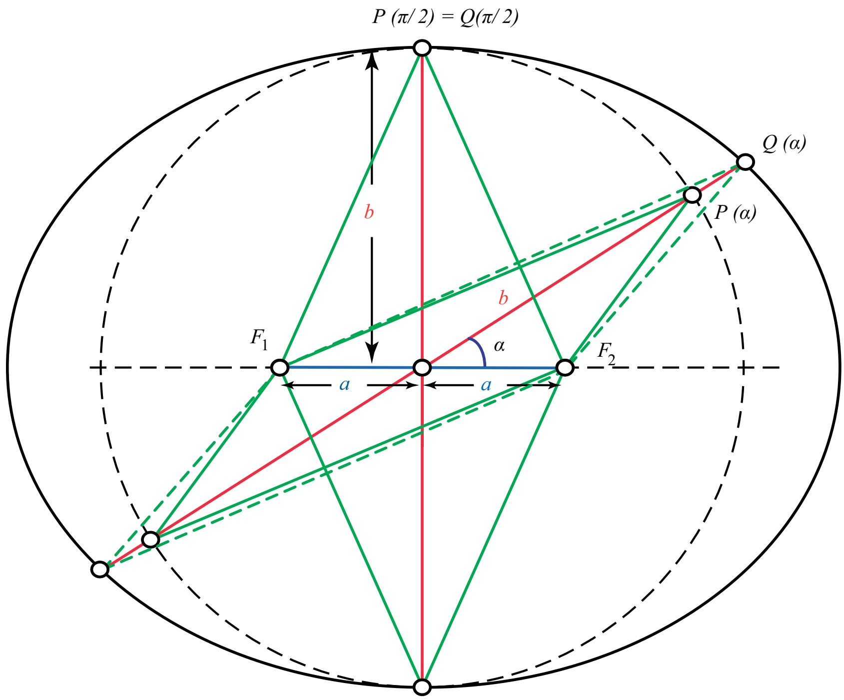
Theorem. *The parallelogram with maximum perimeter for given diagonals is the rhombus.*

Proof.



$$|F_1P(\pi/2)| + |F_2P(\pi/2)| = |F_1Q(\alpha)| + |F_2Q(\alpha)| \geq |F_1P(\alpha)| + |F_2P(\alpha)|.$$

■

Summary. By using the ellipse with foci at the extreme points of the shortest diagonal and the minor axis being the longest diagonal, it is proved without words that the parallelogram with maximum perimeter for given diagonals is the rhombus.

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MSC: Primary 51M15.

Curing Instant Insanity II

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Instant Insanity II is a 4×4 sliding combination puzzle designed by Philip Orbanes. The puzzle is related to both the famous 15 puzzle [**10**] and its namesake, Instant Insanity [**1, 3, 4, 6, 9**]. In Instant Insanity II, there are 16 distinct tiles arranged into four rows and four columns. There is a top row with two empty slots. Tiles may be moved up or down along a column, provided that there is an empty slot. The top row and the bottom row rotate; however, the other rows do not (see Figure 1). Thus, the tiles may be scrambled by using a combination of sliding tiles along a column, rotating the top row, and rotating the bottom row. Normally, the top row is left empty, except when moving one tile to another location on the bottom four rows.

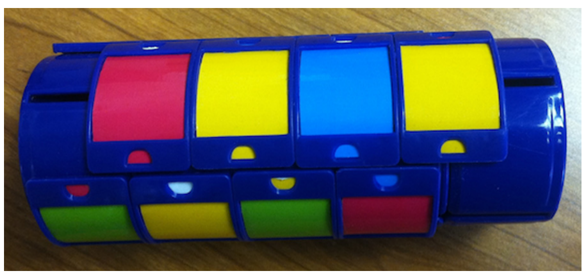

Figure 1 Instant Insanity II puzzle—The two columns showing are (top) wR, bY, wB, rY and (bottom) rG, wY, yG, bR.

Each tile has a large rectangular window in the center, along with two smaller semicircular windows on either side. Each window is colored one of five colors: blue (B), green (G), red (R), white (W), and yellow (Y). In the retail version of Instant Insanity II, the center window and the right window are always the same color. Hence, we denote the color pattern on the tiles with using a lower case letter (for the left window) and an upper case letter (for the center/right window). For example, the tile with a blue left window, green center window, and green right window is denoted 'bG.' The 16 tiles in Instant Insanity II are bG, bR, bW, bY, gB, gR, gW, rB, rG, rY, wB, wR, wY, yB, yG, and yW.

The goal of the puzzle is to scramble the tiles and then to arrange them in such a way that

(i) no color appears twice as the center window on a row or column, and

(ii) on each row, the right window is the same color as the left window of the next tile.

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The manufacturer claims that all combinations of the tiles are possible and that there is a unique solution, up to rotations of the columns and permutations on the rows. However, Richmond and Young [**7**] show that there are in fact two solutions, as shown in Figure 2. These solutions are unique up to permutations on the rows and rotations of the columns.

yB	bG	gW	wY
gR	rB	bY	yG
rY	yW	wB	bR
bW	wR	rG	gB

Solution 1

yB	bG	gR	rY
wR	rB	bY	yW
rG	gW	wB	bR
bW	wY	yG	gB

Solution 2

Figure 2 The two solutions of Instant Insanity II.

The goal of this paper is to explore possible ways of altering the puzzle so that there is a unique solution. Our proposed methods include adding an additional row or column using the existing colors, adding an additional color to create a 5×5 puzzle, and removing a color to create a 3×3 puzzle. Of these, we show that only the 3×3 puzzle yields a unique solution. These methods involve considering the tiles of the puzzle as the elements of a combinatorial design. In an attempt to restrict the puzzle so that only one of the solutions is achievable, we consider a variation on the puzzle that has n rows, k columns, and only one available slot. On such a puzzle, we determine the permutation group of the tiles. Finally, we present several open problems related to our work.

Adding a row or column

How can we modify the puzzle in a natural way so that there is a unique solution? One possible way to do this is to add an additional row (to create a 5×4 puzzle) or to add an additional column (to create a 4×5 puzzle). Observe that each tile in the original version has two different colors. As there are five colors, there are $5 * 4 = 20$ possible tiles. However, only 16 of these possible tiles are included in the original Instant Insanity II. Thus, there are four "missing tiles," namely gY, rW, yR, and wG. In this section, we will use these tiles to add a new row or column to the puzzle. We will then determine the number of solutions to the resulting 5×4 or 4×5 puzzle.

Theorem 1. *There are 12 solutions to the* 5×4 *Instant Insanity II puzzle, up to rotations of the columns and permutations on the rows.*

Proof. In the 5×4 Instant Insanity II, each row of tiles must form a four cycle. In each such four cycle, the center/right window must match the left window on the tile to its right. Fortunately, the four missing tiles naturally form a four cycle, namely wG, gY, yR, rW. The two solutions given in Figure 2 are unique up to rotations of the columns and permutations of the rows. Hence, we can rotate the new cycle in such a way that it can be appended to either solution given in Figure 2. Note that each color is now represented on the same number of tiles. Therefore, any permutation of the colors will also yield an acceptable solution. However, not all of these solutions are unique up to rotations of the columns and permutations of the rows. To eliminate these extraneous solutions, we fix the tile yB into the top left position. Hence, we can assume without loss of generality that any permutation of the colors will leave blue and yellow fixed. Applying the remaining six permutations to the original solutions yields the 12 solutions listed in Figure 3. ■

yB	bG	gW	wY
gR	rB	bY	yG
rY	yW	wB	bR
bW	wR	rG	gB
wG	gY	yR	rW

yB	bR	rW	wY
rG	gB	bY	yR
gY	yW	wB	bG
bW	wG	gR	rB
wR	rY	yG	gW

yB	bW	wG	gY
wR	rB	bY	yW
rY	yG	gB	bR
bG	gR	rW	wB
gW	wY	yR	rG

yB	bG	gR	rY
gW	wB	bY	yG
wY	yR	rB	bW
bR	rW	wG	gB
rG	gY	yW	wR

yB	bR	rG	gY
rW	wB	bY	yR
wY	yG	gB	bW
bG	gW	wR	rB
gR	rY	yW	wG

yB	bW	wR	rY
wG	gB	bY	yW
gY	yR	rB	bG
bR	rG	gW	wB
rW	wY	yG	gR

yB	bG	gR	rY
wR	rB	bY	yW
rG	gW	wB	bR
bW	wY	yG	gB
gY	yR	rW	wG

yB	bR	rG	gY
wG	gB	bY	yW
gR	rW	wB	bG
bW	wY	yR	rB
rY	yG	gW	wR

yB	bW	wR	rY
gR	rB	bY	yG
rW	wG	gB	bR
bG	gY	yW	wB
wY	yR	rG	gW

yB	bG	gW	wY
rW	wB	bY	yR
wG	gR	rB	bW
bR	rY	yG	gB
gY	yW	wR	rG

yB	bR	rW	wY
gW	wB	bY	yG
wR	rG	gB	bW
bG	gY	yR	rB
rY	yW	wG	gR

yB	bW	wG	gY
rG	gB	bY	yR
gW	wR	rB	bG
bR	rY	yW	wB
wY	yG	gR	rW

Figure 3 Solutions of a 5 × 4 Instant Insanity II puzzle.

We now turn our attention to the 4 × 5 puzzle. To do this, we will construct the rows of the puzzle starting with the tile with a blue left window. For purposes of exposition, we will refer to this as a "blue tile."

Theorem 2. *There is no solution to the* 4 × 5 *Instant Insanity II puzzle.*

Proof. Each row of the puzzle must contain a blue tile. Further, each row must be a 5-cycle. We can construct these cycles, beginning with the blue tile using an exhaustive search of all options. For instance, bR must be followed by either rY, rG, or rW. Likewise, rY must be followed by either yW or yG, at which point the only option to follow yW is wG and then gB. This yields the row bR, rY, yW, wG, gB. The same process can be used to construct all possible rows containing the blue tiles. These rows are listed in Figure 4.

To create a solution, we must take one row from each of these sets and rotate it so that each color appears at most once on each row and column. Since rotations on the columns and permutations on the rows are allowed, we can assume that the element of Set 1 is fixed in its rotation.

Suppose that we take the row 1e. Any row from another set that shares a tile with 1e cannot be used. Thus, we can eliminate 2a, 2c, 2d, and 2f from Set 2, 3a, 3d, and 3e from Set 3, and 4b, 4e, and 4f from Set 4. If we choose to fix 2b, then we can further eliminate 3b and 3f, leaving only 3c from Set 3. We can also eliminate 4c and 4d, leaving only 4a from Set 4. However, 3c and 4a share a tile, namely, yB. So there is no possible solution when we fix 2b with 1e. Our only option left from Set 2 is 2e. In Figure 5, we look at the rotations of 2e when 1e is fixed.

In each case, the bold-faced tile will violate the condition that each color appears at most once on each row and column. From this, we see there are no rotations of 2e

Set 1

a)	bR,	rY,	yW,	wG,	gB
b)	bR,	rY,	yG,	gW,	wB
c)	bR,	rG,	gW,	wY,	yB
d)	bR,	rG,	gY,	yW,	wB
e)	bR,	rW,	wY,	yG,	gB
f)	bR,	rW,	wG,	gY,	yB

Set 2

a)	bY,	yW,	wR,	rG,	gB
b)	bY,	yW,	wG,	gR,	rB
c)	bY,	yG,	gR,	rW,	wB
d)	bY,	yG,	gW,	wR,	rB
e)	bY,	yR,	rG,	gW,	wB
f)	bY,	yR,	rW,	wG,	gB

Set 3

a)	bG,	gY,	yR,	rW,	wB
b)	bG,	gY,	yW,	wR,	rB
c)	bG,	gW,	wR,	rY,	yB
d)	bG,	gW,	wY,	yR,	rB
e)	bG,	gR,	rW,	wY,	yB
f)	bG,	gR,	rY,	yW,	wB

Set 4

a)	bW,	wR,	rG,	gY,	yB
b)	bW,	wR,	rY,	yG,	gB
c)	bW,	wG,	gR,	rY,	yB
d)	bW,	wG,	gY,	yR,	yB
e)	bW,	wY,	yG,	gR,	rB
f)	bW,	wY,	yR,	rG,	gB

Figure 4 Possible rows in the 4×5 puzzle.

1e	bR	rW	wY	yG	gB
2e(i)	**yR**	rG	gW	wB	bY
2e(ii)	rG	**gW**	wB	bY	yR
2e(iii)	gW	wB	**bY**	yR	rG
2e(iv)	wB	bY	yR	**rG**	gW
2e(v)	bY	yR	rG	gW	**wB**

Figure 5 The rotations of row 2e versus row 1e.

that we can use when we have 1e fixed. Thus, there are no solutions when we have 1e fixed. Further, since any other element of Set 1 can be obtained from 1e by permuting the colors, it follows that there are no solutions to the 4×5 puzzle. ■

The method of Theorem 2 was also used to confirm Theorem 1.

Adding or removing a color

In the last section, our attempts to "fix" the puzzle by adding an additional row or column using the "missing tiles" were unsuccessful. However, we are steadfast in the belief that there is a "natural fix" of the puzzle that will have a unique solution. Using the original colors did not yield a unique solution. So what happens when we add a color or remove one of the existing colors? In both cases, we wish to preserve the "spirit" of the original puzzle. In the 4×4 puzzle, there are four blue tiles and three tiles of each of the remaining four colors. Thus, we create a 5×5 puzzle by adding a sixth color, purple. This will be represented by a "P." To preserve the spirit of the original puzzle, we will have five blue tiles and four tiles of each of the other five colors. Hence, the tiles we will use for our 5×5 puzzle are: rB, gB, yB, wB, pB, bR, gR, wR, pR, bG, rG, yG, pG, bY, wY, rY, pY, bP, gP, yP, and wP.

Theorem 3. *There is no solution to the* 5×5 *Instant Insanity II puzzle.*

Because of the extensive number of cases involved, we instead sketch a proof of the result. First, note that each row of the puzzle must contain one of the blue tiles and form a 5-cycle. Hence, a similar method to what was described in Theorem 2 can be used to construct the sets given in Figure 6. Finally, perform an exhaustive search of all possible combinations to show that there is no solution to this puzzle.

bR,	rG,	gW,	wY,	yB
bR,	rG,	gW,	wP,	pB
bR,	rG,	gP,	pY,	yB
bR,	rW,	wP,	pY,	yB
bR,	rW,	wP,	pG,	gB
bR,	rW,	wY,	yG,	gB
bR,	rW,	wY,	yP,	pB
bR,	rY,	yG,	gW,	wB
bR,	rY,	yG,	gP,	pB
bR,	rY,	yW,	wP,	pB
bR,	rY,	yP,	pG,	gB

bG,	gR,	rY,	yW,	wB
bG,	gR,	rY,	yP,	pB
bG,	gR,	rW,	wY,	yB
bG,	gR,	rW,	wP,	pB
bG,	gW,	wR,	rY,	yB
bG,	gW,	wY,	yP,	pB
bG,	gW,	wP,	pR,	rB
bG,	gW,	wP,	pY,	yB
bG,	gP,	pR,	rY,	yB
bG,	gP,	pR,	rW,	wB
bG,	gP,	pY,	yW,	wB

bY,	yG,	gR,	rW,	wB
bY,	yG,	gW,	wR,	rB
bY,	yG,	gW,	wP,	pB
bY,	yG,	gP,	pR,	rB
bY,	yW,	wR,	rG,	gB
bY,	yW,	wP,	pR,	rB
bY,	yW,	wP,	pG,	gB
bY,	yP,	pR,	rG,	gB
bY,	yP,	pR,	rW,	wB
bY,	yP,	pG,	gR,	rB
bY,	yP,	pG,	gW,	wB

bW,	wR,	rG,	gP,	pB
bW,	wR,	rY,	yG,	gB
bW,	wR,	rY,	yP,	pB
bW,	wY,	yG,	gR,	rB
bW,	wY,	yG,	gP,	pB
bW,	wY,	yP,	pR,	rB
bW,	wY,	yP,	pG,	gB
bW,	wP,	pG,	gR,	rB
bW,	wP,	pR,	rG,	gB
bW,	wP,	pR,	rY,	yB
bW,	wP,	pY,	yG,	gB

bP,	pR,	rG,	gW,	wB
bP,	pR,	rY,	yG,	gB
bP,	pR,	rY,	yW,	wB
bP,	pR,	rW,	wY,	yB
bP,	pG,	gR,	rY,	yB
bP,	pG,	gR,	rW,	wB
bP,	pG,	gW,	wR,	rB
bP,	pG,	gW,	wY,	yB
bP,	pY,	yG,	gR,	rB
bP,	pY,	yG,	gW,	wB
bP,	pY,	yW,	wR,	rB

Figure 6 The sets for the 5×5 puzzle.

What happens if we remove one of the colors from Instant Insanity II? Again, to preserve the spirit of the original puzzle, we remove any of the colors, except blue. Without loss of generality, suppose that we remove the color white. This leaves us with the tiles gB, rB, yB, bG, yG, bR, gR, bY, and rY. These tiles can be arranged to form a 3×3 puzzle. We invite the reader to prove Theorem 4 given below.

Theorem 4. *The unique solution to the* 3×3 *puzzle is given in Figure 7.*

rB	bG	gR
yG	gB	gY
bR	rY	yB

Figure 7 Solution to the 3×3 puzzle.

The permutation group on the tiles

Throughout this paper, we have attempted to find a variation of Instant Insanity II that only permits a unique solution. These previous attempts have been combinatorial in nature in the sense that we added or removed tiles. We then determined the number of solutions to the resulting puzzle. What if we instead considered an algebraic way of restricting the puzzle? A typical question in modern algebra is "What can we do with a given set of operations?" In other words, can we restrict the mechanics of the puzzle so that only one of the two possible solutions is achievable?

When discussing permutations, it is convenient (and conventional) to write such permutations as the product of disjoint cycles. As usual, we will omit any fixed points in this representation (see, for example, Fraleigh [**2**]). For example, we can obtain Solution 2 from Solution 1 by replacing gW with gR, gR with wR, wR with wY, wY with rY, rY, with rG, rG with yG, yG with yW, and yW with gW. This is represented more compactly in cycle notation as (gW, gR, wR, wY, rY, rG, yG, yW).

We can also write this cycle as a product of transpositions as follows:

$$\begin{aligned}&(\text{gW, gR, wR, wY, rY, rG, yG, yW})\\&\quad= (\text{gW,gR})(\text{gW,wR})(\text{gW,wY})(\text{gW,wY})(\text{gW,rY})(\text{gW,rG})(\text{gW,yG})(\text{gW,yW}).\end{aligned}$$

Notice that this has been written as a product of an odd number of transpositions. Hence, we say that it is an odd permutation. However, any permutation on the rows is an even permutation as is a rotation of the columns. Thus, Solution 2 can only be obtained from Solution 1 using an odd permutation on the tiles. For this reason, if we were able to restrict the permutations on the tiles of Instant Insanity II to a subgroup of the alternating group (i.e., the collection of all even permutations), then we would have a unique solution. One of the most well-known examples of the alternating group is the 15 puzzle. Puzzle master Sam Loyd challenged people to swap tiles 14 and 15 in the 15 puzzle, while returning all other tiles to their original positions. Since the permutation group of the 15 puzzle is the alternating group, the odd permutation (14, 15) is impossible [**10**].

Based on the above comments, we are motivated to find the group of permutations on an Instant Insanity II puzzle. In particular, we generalize this to an Instant Insanity II puzzle with n rows and k columns. For convenience of exposition, we will assume that the tiles on such a puzzle are represented by the numbers $1,\ldots,nk$. Further, assume that in its initial configuration, the tile in the ith row and jth column is $(i-1)k + j$.

Following various treatments of the 15 puzzle (see, for example, [**10**]), it is sufficient to consider only those permutations in which there are no tiles in the top row. Essentially, this allows five different actions on the puzzle. The first is the permutation ρ, which is a rotation of the bottom row of tiles. Similarly, τ rotates the entire puzzle. If we have two slots available, then we can "flip" or transpose any two elements (say 1 and 2) of the top row of tiles by shifting these tiles into the empty row. Then rotate the top row and drop tile 1 into the second column. We then rotate the empty row and drop tile 2 into column 1. Using a similar technique, we can transpose the top two tiles in one column by lifting both elements into the empty row and dropping them down in reverse order. Finally, ν involves moving the entire first column up one slot, rotating the bottom row and the empty row one unit to the right, shifting the second column one unit down, then rotating the bottom row one unit to the left (see Figure 8 for an example of the case where $n = k = 4$). We refer to this as a *vertical shift* on the first column. Using a combination of ν and τ, we can perform a corresponding vertical shift on any column. These five actions act as the generators for the puzzle group. These generators are written as the product of disjoint cycles below:

(i) $\rho = ((n-1)k+1, (n-1)k+2, \ldots, nk)$;
(ii) $\tau = \prod_{i=1}^{n} ((i-1)k+1, (i-1)k+2, \ldots, ik)$;
(iii) $(1, 2)$;
(iv) $(1, k+1)$;
(v) $\nu = (1, k+1, 2k+1, \ldots, (n-1)k+1, (n-2)k+2, (n-3)k+2, \ldots, 2)$.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

→

1			
5	2	3	4
9	6	7	8
13	10	11	12
	14	15	16

→

1			
5	2	3	4
9	6	7	8
13	10	11	12
16		14	15

→

	1		
5	2	3	4
9	6	7	8
13	10	11	12
16		14	15

→

5	1	3	4
9	2	7	8
13	6	11	12
16	10	14	15

→

5	1	3	4
9	2	7	8
13	6	11	12
10	14	15	16

Figure 8 An example of a vertical shift on column 1, where $n = k = 4$.

So, what is the permutation group on an $n \times k$ Instant Insanity puzzle with two slots? Basing our argument on that of Richmond and Young [**7**], it is sufficient to show that all transpositions of distinct tiles are possible. Recall that we are able to transpose any two tiles in the top row. If ℓ is in the first column, then we first switch tiles 1 and 2. Then we lift tile ℓ into position 1 using vertical shifts. We then switch tiles 1 and ℓ. We then reverse the vertical shifts to place tile 1 in position ℓ. We then flip tiles 2 and ℓ to achieve the desired transposition. We encourage the reader to use a similar argument to show that if tile ℓ is not in the first column, then it can also be transposed with tile 1.

Suppose that ℓ and q are any two distinct tiles. From above, we know that we can swap tiles 1 and ℓ and we can swap tiles 1 and q. To swap tiles ℓ and q, we transpose tiles 1 and ℓ. Now, tile ℓ is in position 1, so we can swap it with tile q. Now, tile 1 is in position ℓ, tile ℓ is in position q, and tile q is in position 1. So, we now swap tile q (in position 1) with tile 1 (in position ℓ) to achieve the desired transposition. Hence, every permutation of the tiles is possible. In other words, the group of permutation on the tiles of an Instant Insanity II puzzle with two slots is the symmetric group on nk letters.

But what if one of the two slots were damaged either through defect or through the natural course of play? Is it still possible to achieve every permutation despite only having one slot with which to work? To answer this question, we consider the permutation group $G_{n,k}$ on an Instant Insanity II puzzle with only one slot. Since we only have one slot, we may not have $(1, k+1)$ and $(1, 2)$ as natural generators. Thus, we assume that the only natural generators are ρ, τ, and ν as described above. With these generators in mind, we now proceed to determine the puzzle group $G_{n,k}$.

Theorem 5. *For the group $G_{n,k}$ of permutations on an $n \times k$ Instant Insanity II puzzle with one slot, we have the following:*

(i) *If $n = 1$ or $k = 1$, then $G_{n,k}$ is isomorphic to $\mathbb{Z}_k$, the cyclic group on k letters.*
(ii) *If $n \geq 2$, $k \geq 3$, and k is odd, then $G_{n,k}$ is isomorphic to A_{nk}, the alternating group on nk letters.*
(iii) *If $n \geq 2$ and k is even, then $G_{n,k}$ is isomorphic to S_{nk}, the symmetric group on nk letters.*

Proof. If $k = 1$, then no moves are possible. Hence, $G_{n,1}$ is the trivial group. If $n = 1$, then the only possible moves are rotations of the single row. Thus, $G_{n,k}$ is isomorphic to the cyclic group on k letters in either case.

If $n \geq 2$ and $k = 2$, then any vertical shift will leave one element fixed on the bottom row. Thus, if we wanted to interchange two tiles, then we would use a series of vertical shifts to put them on the bottom row. We would then rotate the bottom row. The series of vertical shifts could then be inverted to interchange the two tiles, while returning all other tiles to their original positions. Since any transposition is possible, it follows that $G_{n,2}$ is isomorphic to S_{2n}.

Thus, we can assume that $k \geq 3$ and $n \geq 2$. We now show that A_{nk} is a subgroup of $G_{n,k}$. The alternating group is generated by the set of 3-cycles of the form $(1, 2, m)$, where $m \notin \{1, 2\}$. We will show that any such 3-cycle is contained in $G_{n,k}$.

We begin by showing the cycle $(1, 2, 3) \in G_{n,k}$. We will denote the columns that tiles 1, 2, and 3 are in when the puzzle is in its original state as columns "a,""b," and "c,," respectively. We perform a series of vertical shifts on columns a, b, and c until tiles 1, 2, and 3 are in the bottom row. Note that when applying the vertical shifts, these three tiles remain in their respective columns. If $k = 3$, then we can cycle these three elements using the bottom row rotation. We then apply the inverses of the vertical shifts to get the tiles back to the top row. In all other cases, we rotate the bottom row until we get tile 1 in column b. Then, we apply the inverse vertical shift on column b to bring tile 1 to the top of the column. We then rotate the bottom row until tile 2 is in column c. Tile 2 is then brought to the top of column c using inverse vertical shifts. Finally, we rotate the bottom row until tile 3 is in column a and apply inverse vertical shifts. Note that, in each case, we can apply the vertical shifts in such a way that we only move the desired tile on the bottom row.

We now show that for $m \notin \{1, 2\}$ the 3-cycle $(1, 2, m)$ is in $G_{n,k}$ using a similar process. Move the tile m to the bottom row using the appropriate vertical shifts. Note that, when doing a vertical shift, only one tile on the bottom row gets moved. Rotate the bottom row to put tile m in column 'c.' If m was in the first two columns, then we perform inverse vertical shifts to return tile 1 or 2 to the top of their column. Then we apply a series of vertical shifts to get tile m to the top of column c. We now perform the permutation $(1, 2, 3)$ to rotate the tiles. We then invert the permutation used to bring tile m to the top row. This results in either $(1, 2, m)$ or $(1, 2, m)$ together with a set of disjoint transpositions. In the latter case, we square the permutation. In either case, we have the required 3-cycle. Since $(1, 2, m) \in G_{n,k}$ for all $m \notin \{1, 2\}$, we have that A_{nk} is a subgroup of $G_{n,k}$.

Recall that our generators for $G_{n,k}$ consist of a k-cycle, a product of n k-cycles, and a $(2n - 1)$-cycle. Thus, if k is odd, then all of our generators are even permutations. This implies that $G_{n,k}$ is a subgroup of A_{nk}. From above, we have that $G_{n,k}$ is isomorphic to A_{nk}.

Suppose k is even. This implies that we have at least one odd permutation, namely the rotation on the bottom row. By Cayley's theorem, $G_{n,k}$ is a subgroup of S_{nk}. Therefore, the order of $G_{n,k}$ divides $(nk)!$ by Lagrange's theorem. Further, we showed above that A_{nk} is a subgroup of $G_{n,k}$. Ergo, $|G_{n,k}| \geq |A_{nk}| = \frac{(nk)!}{2}$. Further, because $G_{n,k}$ has at least one odd permutation, $|G_{n,k}| \geq \frac{(nk)!}{2} + 1$. However, there are no integers between $\frac{(nk)!}{2} + 1$ and $(nk)!$ that divide $(nk)!$ other than $(nk)!$ itself. Therefore, $|G_{n,k}| = (nk)!$, which implies that $G_{n,k}$ is isomorphic to S_{nk}. ∎

From this, it follows that the removal of one of our available slots does not change the group of permutations on the tiles of the 4×4 Instant Insanity II puzzle. It is true that such a puzzle would likely be more difficult. In particular, we would expect that the length of an average "word" would be longer using the smaller generating set. For example, one of our natural generators for the retail puzzle is $(1, 5)$. Using the computer algebra system Sage, we find that the shortest way to express this as a product of the above generators would be:

$$(1,5) = \rho^{-1}\nu^{-1}\rho^{-1}\nu\rho^{-1}\nu^{-1}\rho^{-1}\nu\rho^{2}\nu^{-2}\rho\nu^{-1}\rho^{-1}\nu^{6}\rho\nu^{-2}\rho^{-1}\nu^{-1}\rho\nu^{-1}.$$

What if both slots in the top row were damaged and unusable? In this case, we can only rotate the entire puzzle or rotate the bottom row. From this it follows that the group of permutations on an $n \times k$ Instant Insanity II puzzle with no slots is isomorphic to a direct product of $\mathbb{Z}_k$ with itself.

In the previous sections, we considered 5×4, 4×5, 5×5, and 3×3 puzzles. The only one of these cases that had multiple solutions was the 5×4 puzzle. Since the number of columns in this case is even, the permutation group on the tiles will be S_{20}. This holds true regardless if we have a puzzle with one slot or two. Hence, restricting any of the previously considered configurations to a puzzle with one slot does not affect the number of solutions.

Open problems

We end this paper by describing three open problems about puzzles like Instant Insanity II.

There has been much research devoted to determining the minimum number of moves necessary to solve certain mechanical puzzles, regardless of how scrambled. This is called *God's number*. For example, it is known that God's number for the Rubik's Cube is 20 [**8**]. Thus, a natural problem is to determine God's number for Instant Insanity II.

For our second problem, we note that the solved state of Instant Insanity II is similar to several combinatorial designs (see [**11**] for more information on combinatorial designs). We define an (n, k, c)-II2 design to be an $n \times k$ grid, where each square on the grid is assigned an ordered pair (x, y) where x and y are each one of c possible colors and $x \neq y$. The ordered pairs must be assigned in such a way that:

(i) The second coordinate does not appear twice in any row or column of the grid.

(ii) When the grid is considered as a cylinder, the second coordinate matches the first coordinate of the following tile in the same row.

(iii) No two squares on the grid are assigned the same ordered pair.

Under what conditions does an (n, k, c)-II2 design exist? If such a design exists, then how many solutions are possible? Under what conditions does a unique solution exist, up to rotation of the columns and permutations of the rows? In particular, given n and k, what is the minimum c such that a solution exists? For what values of n does an $(n, n, n + 1)$-II2 design exist?

Finally, what are ways to mathematically quantify the difficulty of a puzzle such as Instant Insanity II? A necessary condition for a challenging puzzle would be a large number of possible states. However, a large number of states does not guarantee that the puzzle will be difficult. For example, the Tower of Hanoi with n disks has 3^n possible states [**5**]. However, having more disks does not appreciably increase the difficulty of the puzzle, provided that you know the algorithm. A second possible way to quantify the difficulty of a puzzle is the average (or maximum) word length, based on our natural generators. For example, we expect that the one slot Instant Insanity II puzzle would be more difficult than the two slot version for this reason.

Acknowledgment The authors wish to acknowledge the anonymous referees for their helpful comments regarding the final version of this paper.

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Summary. Instant Insanity II is a 4 by 4 sliding tile puzzle designed by Philip Orbanes. The packaging indicates that there is a unique solution to the puzzle, up to rotations of the columns and permutations on the rows. However, a recent paper by Richmond and Young shows that there are in fact two solutions to the puzzle. This paper presents several attempts at “fixing” the puzzle to guarantee a unique solution. Of these, the only one that guaranteed a unique solution was removing a color to create a 3 by 3 puzzle.

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Proof Without Words: An Identity for a Recurrence Satisfied by the Fibonacci and Lucas Numbers

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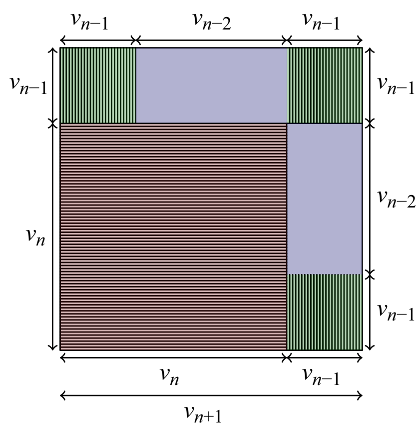
Theorem. $v_{n+1} = v_n + v_{n-1} \Longrightarrow v_{n+1}^2 = v_n^2 + 3v_{n-1}^2 + 2v_{n-1}v_{n-2}$.

Proof.



■

Notice that the theorem applies to Fibonacci numbers ($F_{n+1} = F_n + F_{n-1}$, $F_0 = 0$, $F_1 = 1$) and Lucas numbers ($L_{n+1} = L_n + L_{n-1}$, $L_0 = 1$, $L_1 = 3$), since both follow the same recurrence relation.

Acknowledgment. The author wants to thank the referees and the editor for their comments to former versions of this proof without words which led to a completely new presentation.

Summary. We present a visual proof of an identity for three consecutive terms of a recurrence relation satisfied by the Fibonacci and Lucas numbers.

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Extreme Wild Card Poker

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The mathematics of poker has been studied extensively by mathematicians and gambling enthusiasts alike. In this paper we will restrict ourselves to a five-card stud game where each player receives five cards from the deck. Our deck will consist of a standard 52-card deck, consisting of four suits {♣, ♢, ♡, ♠} each with 13 denominations {2, 3, . . . , 10, J,Q,K,A}, to which w wild cards have been added. We fix the hierarchy of poker hands from best to worst in the order five-of-a-kind, straight flush, four-of-a-kind, full house, flush, straight, three-of-a-kind, two pair, one pair, and junk*. Aces are allowed to be high or low. When aces are only allowed to be high, there are only minor changes to just a few of the calculations. We do not and need not distinguish royal flushes, for a royal flush is simply the highest ranking straight flush, just as five aces is the highest ranking five-of-a-kind. When a standard 52-card deck is used poker hand rankings follow directly from the relative frequencies with which the hands occur. For instance, there are 40 straight flushes in a standard deck and 624 four-of-a-kind hands. Thus straight flushes are ranked higher than four-of-a-kind hands. With the addition of wild cards (cards that can be designated anything the holder wishes), the standard rankings of hands are no longer consistent. See [**2**] for detailed explanations of some of the paradoxes that can arise from the addition of wild cards, and [**1**] which proposes an alternate ranking scheme to address those paradoxes.

For the purposes of this note however, the authors wondered what might happen if the standard ranking of poker hands is maintained and the addition of wild cards is taken to the extreme. In particular, we wanted to answer two specific questions:

1. How many wild cards would need to be added to a standard deck to make five-of-a-kind the most common hand?
2. How many wild cards would be needed to make the chances of drawing a five-of-a-kind better than 50%?

Calculations

The frequencies of poker hands using a standard 52-card deck are well known; see [**3**] for an introduction to the common types of computations needed. The calculations when using wild cards are done in a similar manner, keeping in mind that the number

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*For those unfamiliar with the nonintuitive hands, a straight flush consists of five cards in a row, all of the same suit, a full house is three-of-a-kind together with one pair of a different rank, a flush is five cards of the same suit, and a straight is five cards in a row. Five-of-a-kind is of course not possible without the addition of wild cards, to be described momentarily.

of hands of any given type depends on two variables: the number of wild cards in the hand (from 0 to 5) and the total number of wild cards added to the deck. As an example, let $5K(i, w)$ be the number of five-of-a-kinds in a deck with w added wild cards and i wild cards in the hand. Using this notation then, the total number of five-of-a-kind hands in a deck with w added wild cards would be $\sum_{i=0}^{5} 5K(i, w)$, where $5K(0, w) = 0$; $5K(i, w) = \binom{13}{1}\binom{4}{4-(i-1)}\binom{w}{i}$ for $i = 1, 2, 3, 4$; and $5K(5, w) = \binom{w}{5}$. The formulas for $5K(0, w)$ and $5K(5, w)$ are straightforward: five-of-a-kind with no wild cards is impossible, and the number of hands that are comprised solely of five wild cards is just $\binom{w}{5}$. For the remaining formula, when $i = 1$ for example, choose one of the 13 denominations, $\binom{13}{1}$ ways, choose all four suits of that rank, $\binom{4}{4}$ ways, and then choose one wild card, $\binom{w}{1}$ ways.

TABLE 1: General Formula for Poker Hands

Poker Hands	Formulas for i Wild Cards in Hand, w Wild Cards Added to Deck
Five-of-a-Kind	$5K(i, w) = \binom{13}{1}\binom{4}{4-(i-1)}\binom{w}{i}$ for $i = 1, 2, 3, 4$
	$5K(5, w) = \binom{w}{5}$
Straight Flush	$SF(0, w) = \binom{10}{1}\binom{4}{1} = 40$
	$SF(i, w) = \left[\binom{10}{1}\binom{5}{i} - 9\binom{4}{i-1}\right]\binom{4}{1}\binom{w}{i}$ for $i = 1, 2, 3$
Four-of-a-Kind	$4K(i, w) = \binom{13}{1}\binom{4}{4-i}\binom{12}{1}\binom{4}{1}\binom{w}{i}$ for $i = 0, 1, 2$
	$4K(3, w) = \binom{13}{2}\binom{4}{1}^2\binom{w}{3} - SF(3, w)$
Full House	$FH(i, w) = \binom{13}{i+1}\binom{4}{3-i}\binom{12}{1-i}\binom{4}{2}\binom{w}{i}$ for $i = 0, 1$
Flush	$F(0, w) = \binom{13}{5}\binom{4}{1} - SF(0, w) = 5108$
	$F(i, w) = \binom{13}{5-i}\binom{4}{1}\binom{w}{i} - SF(i, w)$ for $i = 1, 2$
Straight	$S(0, w) = \binom{10}{1}\binom{4}{1}^5 - SF(0, w) = 10{,}200$
	$S(i, w) = \left[\binom{10}{1}\binom{4}{i} + \binom{4}{i-1}\right]\binom{4}{1}^{5-i}\binom{w}{i} - SF(i, w)$ for $i = 1, 2$
Three-of-a-Kind	$3K(i, w) = \binom{13}{1}\binom{4}{3-i}\binom{12}{2}\binom{4}{1}^2\binom{w}{i}$ for $i = 0, 1$
	$3K(2, w) = \binom{13}{3}\binom{4}{1}^3\binom{w}{2} - SF(2, w) - F(2, w) - S(2, w)$
Two Pair	$2P(0, w) = \binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1} = 123{,}552$
One Pair	$1P(0, w) = \binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3 = 1{,}098{,}540$
	$1P(1, w) = \binom{13}{4}\binom{4}{1}^4\binom{w}{i} - SF(1, w) - F(1, w) - S(1, w)$
Junk	$\binom{13}{5}\binom{4}{1}^5 - SF(0, w) - S(0, w) - F(0, w) = 1{,}302{,}540$
TOTAL:	$\binom{52+w}{5}$

For each hand type, a similar formula can be constructed and is illustrated in Table 1. For straights, it is important to remember that there are ten straights, the lowest being {A, 2, 3, 4, 5} and the highest being {10, J, Q, K, A}.

There are several interesting things worthy of mention in Table 1. First, the number of junk hands and the number of two pair hands do not increase with the number of wild cards. Any junk hand that contains a wild card will automatically be "upgraded" to at least a one pair hand. Similarly, if W is a wild card, then for example

$\{W, 4\diamondsuit, 8\diamondsuit, 8\spadesuit, K\clubsuit\}$ is a two pair hand but can also be classified as a three-of-a-kind hand. The total number of two pair hands when $w = 1$ is 123,552 while the total number of three-of-a-kind hands is 137,280. Thus even though three-of-a-kind is ranked higher than two pair, there are more ways to get three-of-a-kind with a two wild card, 54-card deck. But as is noted in [**2**], if a resolution is attempted by changing the ranking so that two pair is ranked higher than three-of-a-kind, then there are more ways to get two pair, because a large number of hands can be classified either way. This unresolvable paradox also arises with one pair and junk. While it is noted in [**2**] that with two wild cards, the number of ways to get a full house exactly equals the number of ways to get four-of-a-kind, that seems to be just an interesting coincidence, not a paradox on the level of the two just noted.

The last point to consider in Table 1 is regarding five-of-a-kind hands. Five-of-a-kind is the only hand that can be created when there are four or five wild cards in the hand. As a consequence of this, the number of five-of-a-kind hands increases more rapidly than any other poker hand as the number of wild cards increase. In fact, since the binomial coefficient $\binom{w}{i}$ can be viewed as a polynomial in w, the frequency of five-of-a-kind hands is the only quintic in w and thus will inevitably become larger than the frequencies of all other hands. For example with $w = 15$, or a 67-card deck, there are 115,128 five-of-a-kind hands and only 113,100 straight flushes, making five-of-a-kind more common than a straight flush.

TABLE 2: Poker Hand Frequencies and Probabilities in 125- and 162-Card Decks

	73 Wild Cards, 125-card deck		110 Wild Cards, 162-card deck	
Hands	Frequency	Probability	Frequency	Probability
Five-of-a-Kind	76607587	0.326641242	439744272	0.503345688
Straight Flush	12128844	0.051715252	41263680	0.047231759
Four-of-a-Kind	76198608	0.324897428	252352944	0.288851440
Full House	208728	0.000889979	312624	0.000357840
Flush	2535580	0.010811266	5625228	0.006438820
Straight	10855956	0.046287882	24167520	0.027662934
Three-of-a-Kind	41072736	0.175126904	88968792	0.101836592
Two Pair	123552	0.000526804	123552	0.000141422
One Pair	13497144	0.057549442	19781520	0.022642575
Junk	1302540	0.005553801	1302540	0.001490930
TOTAL:	234531275	1	873642672	1

At this point, we can now determine the answer to our first question. With the addition of 73 wild cards, creating a 125-card deck, five-of-a-kind becomes the most common hand in poker. But when $w = 72$, five-of-a-kind and four-of-a-kind have respective probabilities 0.3210 and 0.3251.

For our second question, with 110 wild cards added, creating a 162-card deck, there are now 439,744,272 five-of-a-kinds out of the possible $\binom{162}{5} = 873{,}642{,}672$ hands resulting in a 50.33% chance of being dealt a five-of-a-kind. But when $w = 109$, five-of-a-kind has probability 0.4994. The frequencies and probabilities of each type of hand for each of these extreme cases is given in Table 2. So, if you are playing poker with 110 wild cards and you aren't dealt a five-of-a-kind, fold!

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Summary. Rankings of hands in traditional five-card poker are based on the relative frequency of each type of hand occurring. When wild cards are added to the standard deck, these rankings can go awry quite quickly. In this paper, wild card poker is taken to the extreme. We find the minimum number of wild cards needed to ensure five-of-a-kind is the most common hand.

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Artist Spotlight
Dick Termes

Platonic Relationship, Dick Termes; 36-inch painted sphere, 1993. This is a sphere that plays with the five platonic solids and shows their interrelated geometries. The squares, triangles, and pentagons are all turned into cylinder loops.

See interview on page 290.

Proof Without Words: The Automorphism Group of the Petersen Graph Is Isomorphic to S_5

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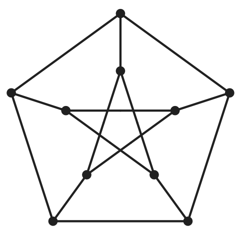

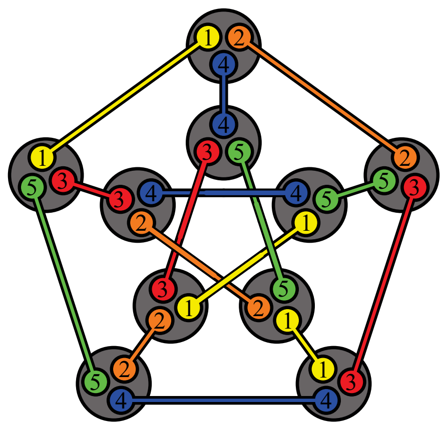

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Summary. The automorphism group of the Petersen graph is known to be isomorphic to the symmetric group on 5 elements. This proof without words provides an insightful and colorful image that proves this fact, without words. The image represents the Petersen graph with the ten 3-element subsets of $\{1, 2, 3, 4, 5\}$ as vertices, and two vertices are adjacent when they have precisely one element in common. This representation of the Petersen Graph is similar to the Kneser graph $KG_{5,2}$, a nice picture of which can be found in John Baez's Visual Insight blog.

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MSC: Primary 20B25; Secondary 05C25.

Fuzzy Knights and Knaves

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On the island of knights and knaves, knights only make true statements and knaves only make false statements. The two tribes are visually indistinguishable, and everyone on the island belongs to exactly one of the tribes. For visitors to the island, this means the only way to determine who is who is to work out the logical consequences of the often cryptic statements made by the natives.

A typical scenario might unfold like this:

Problem 1. *One day you meet Alex, Beth, and Chad, who make the following statements:*

Alex : *There are exactly two knights among us.*

Beth : *If that is true, then I am one of the knights.*

Chad : *But it isn't true, since Alex is a knave.*

Who are the knights and who are the knaves?

Solution: If Alex is a knave, then Chad's statement is true, implying that Chad is a knight. However, the antecedent of Beth's conditional would now be false (since Alex's statement must be false in this scenario). This implies her whole statement is true, meaning she is a knight as well. But then there really are two knights among them, implying that Alex's statement is true after all. This is a contradiction.

It follows that Alex must be a knight. Chad's statement is then false, meaning he is a knave. Since Alex spoke truthfully we conclude there must really be two knights among them, implying that Beth is a knight. □

Puzzles of this sort were developed into a high art by Raymond Smullyan in numerous books [**11, 12, 13, 14**]. His intent was only partly recreational. He also used such puzzles as a device for introducing important ideas in logic, such as Gödel's incompleteness theorem.

Logic is especially interesting in that at least three different disciplines—mathematics, philosophy, and computer science—take a serious interest in it [**6**]. Moreover, each discipline brings its own perspective. One thing that has become clear is that classical logic, by which I mean the sort of logic we typically learn about in math classes, is hardly the end of the story. A modern view of the subject treats classical logic as just one system among many, with the criterion for deciding among them being usefulness, as opposed to correctness. This is analogous to the status of the Euclidean system within geometry generally. (Eklund [**1**] provides a lucid discussion of the issues surrounding this claim.)

Given the perennial popularity of logic puzzles, and the ubiquity of nonclassical logics in modern scholarly discourse, perhaps we should give some thought to nonclassical logic puzzles.

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Classical logic is based on certain principles that seem so intuitive and obvious that our first reaction, upon hearing that someone has presumed to deny them, could well be to get angry. At the heart of the classical system is the *Principle of Bivalence*, by which we mean that there are exactly two truth values: True and False. Two further principles are the *Law of Noncontradiction* and the *Law of the Excluded Middle*. These state, respectively, that the two truth values are mutually exclusive and jointly exhaustive. A nonclassical logic can be thought of informally as any system that rejects one or more of these principles. Readable introductions to this subject can be found in the books by Haack [**2**] and Priest [**8**].

Why might we wish to discard any of these principles? One reason is that a strict dichotomy between true and false, coupled with the requirement that all propositions be assigned at least one of those truth values, runs into the problem of vagueness. Suppose you say of a young teenager, "He is a child." Since a teenager is in some ways like a child and in some ways like an adult, it is problematic to assign this statement a definitive truth value.

We could evade this problem by rejecting the principle of bivalence. Perhaps what is needed are additional truth values, so that we can accept the fact that in some cases truth is a matter of degree. In previous work we investigated knight/knave dialogs for one system of three-valued logic [**10**]. Here we explore "fuzzy logic," in which an infinity of truth values are permitted. Following Smullyan's example, I offer the following puzzles partly for their recreational value, and partly to bring to light important issues in logic.

Fuzzy islands

On classical islands, being a knight or a knave is purely a function of the sort of statements you make. Let us imagine a more complex island, in which knighthood and knavehood are conferred based on a variety of social, cultural, and physical factors. The sociological minutiae need not detain us here. What matters is that on such an island, it would be meaningful to think of someone as being partly a knight and partly a knave. We will continue to suppose that knights only make true statements and knaves only make false statements. However, these traits no longer define what knights and knaves are.

Imagine that on one such island it was discovered that knighthood and knavehood were not permanent conditions. Rather, throughout their lives the islanders cycled between the two states. They would be knights for a while, and then enter a transitional phase in which they were partly knights and partly knaves. Eventually they emerged from the transitional phase as knaves. Then they would persist in this state for a while, until they once more entered the transitional phase and emerged on the other side as knights.

The island's logicians decided it was too constraining to have only two truth values. In their view, truth was best viewed as a matter of degree. The idea was that the statement "He is a knight" is more true when directed at a knight who had just entered his transitional phase than when directed at a knave who had just entered his. For that reason, they employed a continuum of truth values. A truth value, for them, was a real number between 0 and 1 inclusive. A truth value of 0 indicated complete falsity, while 1 indicated complete truth. If the statement, "Joe is a knight" was assigned a truth value of, say, 0.9, then that would indicate that Joe was recently a knight, but was now in the early stages of his transition (so that he was mostly knight and just a little bit knave.)

We should note that it was not just any old statement that could be assigned a truth value other than 0 or 1. For nonvague statements, truth and falsity continued to mean what they have always meant. Intermediate truth values were used only in the case of ambiguous statements. Specifically, they applied only to assignments of knighthood or knavehood to those in the transitional state (or to compound statements built from those atoms through the standard connectives.)

If we let P and Q denote arbitrary propositions, and if we denote by $v(\mathsf{P})$ and $v(\mathsf{Q})$ their truth values, then the islanders employed the following conventions for assigning truth values to compound statements:

$$v(\neg \mathsf{P}) = 1 - v(\mathsf{P})$$
$$v(\mathsf{P} \wedge \mathsf{Q}) = \min(v(\mathsf{P}), v(\mathsf{Q}))$$
$$v(\mathsf{P} \vee \mathsf{Q}) = \max(v(\mathsf{P}), v(\mathsf{Q})).$$

In words, the truth values assigned to a statement and its negation must sum to one. A conjunction is as true as its least true conjunct, while a disjunction is as true as its most true disjunct. All three of these conventions agree with the dictates of classical logic when only classical truth values (1 or 0) are employed.

The convention for conditionals was more complex:

$$v(\mathsf{P} \to \mathsf{Q}) = \begin{cases} 1 & \text{if } v(\mathsf{P}) \leq v(\mathsf{Q}) \\ 1 - (v(\mathsf{P}) - v(\mathsf{Q})) & \text{if } v(\mathsf{P}) > v(\mathsf{Q}). \end{cases}$$

Alternatively, you can view this convention as saying that

$$v(\mathsf{P} \to \mathsf{Q}) = \min(1, 1 - (v(\mathsf{P}) - v(\mathsf{Q})).$$

The idea is that if P is less true than Q, then we declare the conditional $\mathsf{P} \to \mathsf{Q}$ to be true (have truth value 1). This is in agreement with the rule in classical logic. If P is more true than Q, then we view the conditional as defective in some sense. The extent to which it differs from perfect truth is found by subtracting from 1 the magnitude of the drop in truth value from P to Q. This implies a conditional statement whose antecedent is far more true than its conclusion should be regarded as mostly false.

The transitional phase also complicated the island's sociology. A recent knight who was less than halfway through his transitional phase was referred to as a "quasiknight." At this stage he only made statements with high truth values. Near the end of his transition he became a "quasiknave." In this condition he only made statements with low truth values. The process reverses itself when a knave enters his transitional phase. Less than halfway through his transition he is a quasiknave. When he is more than halfway through he becomes a quasiknight.

Specifically, if P is a proposition spoken by a person A, then we have

$$\begin{cases} v(\mathsf{P}) = 1 & \text{if A is a knight.} \\ 0.5 < v(\mathsf{P}) < 1 & \text{if A is a quasiknight.} \\ 0 < v(\mathsf{P}) < 0.5 & \text{if A is a quasiknave.} \\ v(\mathsf{P}) = 0 & \text{if A is a knave.} \end{cases}$$

Moreover, if P is the proposition, "A is a knight," then the above four cases again provide the possible values for $v(\mathsf{P})$. You should note that no islander ever makes statements whose truth value is exactly 0.5. (This restriction is made simply because it leads to cleaner puzzles. In principle a statement could have a truth value of exactly 0.5, it is just that no one on the island would ever utter such a statement.)

These conventions can take some getting used to. Let us pause to make explicit a few principles that will be useful for the problems to come.

- While the islanders are content to accept some ambiguity in assignments of knighthood and knavehood, we, as people looking in on the island from outside, will strive for complete clarity. For that reason, we shall use the expressions “fully knight” and “fully knave” to describe islanders who, respectively, only make statements with truth values 1 and 0.
- The propositions, “Joe is not a knight,” and “Joe is a knave,” should be understood as logically equivalent. They always have the same truth value. For that reason, we can say that if P is the proposition, “Joe is a knight,” then $\neg\mathsf{P}$ is the proposition, “Joe is a knave.” The same applies to the pair of statements, “Joe is not a knave,” and “Joe is a knight.”
- In reasoning through the problems to come, it will help to think of a quasiknight as someone who is more knight than knave. Likewise, you should think of a quasiknave as someone who is more knave than knight.
- There is no vagueness regarding one’s status as a quasiknight or a quasiknave. That is, the propositions “Joe is a quasiknight,” and “Joe is a quasiknave,” can only have truth values 0 or 1. If Joe was recently a knight but has just entered his transitional phase, then the statement “Joe is a quasiknight” has truth value 1, but the statement “Joe is a knight” is vague, and has a truth value somewhere strictly between 0.5 and 1. The same is true if the word “knight” is replaced with “knave,” except that now the statement, “Joe is a knight,” has a truth value strictly between 0 and 0.5.

Warm-up problems

It is only the heartiest of travelers who would converse with the islanders in the hope of gaining information. To appreciate the difficulties, let us try some warm-ups. It will become easier to follow the conventions outlined above if we see how they play out in concrete cases.

Problem 2. *What can you conclude from the following dialog?*

$$\begin{aligned} \textit{Dave}&: \textit{Evan is a quasiknight.}\\ \textit{Evan}&: \textit{Dave is a quasiknave.}\end{aligned}$$

Solution: Both Dave and Evan have made statements that can only have truth values 0 or 1. It follows that each is either fully knight or fully knave. That implies that Evan is not a quasiknight and Dave is not a quasiknave. We conclude that Dave and Evan have both made statements with truth value 0, which implies that both are fully knave. □

Problem 3. *What can you conclude from the following dialog?*

$$\begin{aligned} \textit{Fran}&: \textit{Gina is a knight.}\\ \textit{Gina}&: \textit{Fran's statement has truth value } 0.8.\end{aligned}$$

Solution: Gina’s statement is of a sort that can only have truth value 1 or 0. This is because Fran’s statement has a definite truth value, and that value is either 0.8 or it is not. If we suppose that Gina’s statement has truth value 1, then she must be fully knight. It would then follow that Fran’s statement has truth value 1, which would make her fully knight as well. Since people who are fully knight only make statements with truth value 1, this would imply that Gina's statement actually has truth value 0, which is a contradiction.

It follows that Gina's statement has truth value 0, and therefore that she is fully knave. This implies that Fran's statement also has truth value 0, meaning that she is fully knave as well. So Fran and Gina are both fully knave. □

Problem 4. *Can you think of a simple sentence that no one on the island can say?*

Solution: On a classical island, no one can say "I am a knave," since a knight who said that would be lying, while a knave who said that would be telling the truth. On fuzzy islands it is still the case that no one can make that statement. To see this, let P be the statement "I am a knave." Then $\neg\mathsf{P}$ is the statement "I am a knight."

If the speaker is fully knight or fully knave, then our reasoning would proceed just as it would on a classical island.

If the speaker is a quasiknight, then we have

$$v(\mathsf{P}) = 1 - v(\neg\mathsf{P}) < 1 - 0.5 = 0.5.$$

That is too low a truth value for a quasiknight.

If the speaker is a quasiknave, then a similar calculation shows that $v(\mathsf{P}) > 0.5$. That is too high for a quasiknave. □

Let us conclude this section with two puzzles involving conjunctions and disjunctions.

Problem 5. *Suppose I tell you that my friend Hyla, an islander, said, "I am a knave or a quasiknight." What would you conclude?*

Solution: We let P be the proposition, "Hyla is a knave," and we let Q be the proposition, "Hyla is a quasiknight." Then Hyla's statement is equivalent to $\mathsf{P} \vee \mathsf{Q}$.

Now, $v(\mathsf{Q}) = 0$ or 1. If $v(\mathsf{Q}) = 1$, then Hyla really is a quasiknight. But then we would have

$$v(\mathsf{P} \vee \mathsf{Q}) = \max(v(\mathsf{P}), v(\mathsf{Q})) = 1,$$

which implies that Hyla is fully knight. This is a contradiction.

It follows that $v(\mathsf{Q}) = 0$ and therefore that $v(\mathsf{P} \vee \mathsf{Q}) = v(\mathsf{P})$. If Hyla is fully knight, then $v(\mathsf{P}) = 0$. This is a contradiction, since it entails that someone who is fully knight has made a statement with truth value 0. We get a similar contradiction if we suppose Hyla to be fully knave.

The trouble is that we also get a contradiction if we suppose Hyla to be a quasiknave. For then we would have that $0.5 < v(\mathsf{P}) < 1$ since a quasiknave is more knave than knight. But this would imply that $v(\mathsf{P} \vee \mathsf{Q}) > 0.5$, which is too high for a quasiknave.

So you would conclude that I had lied to you, since no islander can make the statement I attributed to Hyla. (And this, incidentally, provides another possible answer to Problem 4.) □

In our final warm-up problem, it will be convenient to refer to the "rank" of an islander. The idea is that the closer someone is to being fully knight, the higher their rank. Someone who is fully knight is of higher rank than a quasiknight, who in turn is of higher rank than a quasiknave. Someone who is fully knave is of the lowest rank.

Problem 6. *What can you conclude from the following dialog?*

Ivan : *Kate is a knave.*

Jill : *Kate is a knave and Ivan is a knight.*

Kate : *Ivan is of lower rank than Jill.*

Solution: Let us define P to be the proposition, "Kate is a knave," and define Q to be the proposition, "Ivan is a knight." Since

$$v(\mathsf{P} \wedge \mathsf{Q}) = \min(v(\mathsf{P}), v(\mathsf{Q})) \leq v(\mathsf{P}),$$

we see that the truth value of Ivan's statement is not smaller than the truth value of Jill's statement. It follows that Ivan's rank is not lower than Jill's rank, which implies that Kate's statement has truth value 0. Therefore, Kate is fully knave. From this we quickly see that both Ivan's statement and Jill's statement have truth value 1.

So the solution is that Ivan and Jill are fully knight, while Kate is fully knave. □

As an aside, we should mention that if, in Problem 5, we change Jill's statement to, "Kate is a knave or Ivan is a knight," and change Kate's statement to, ''Ivan is of higher rank than Jill," then the solution would be essentially unchanged. In this case we would have

$$v(\mathsf{P} \vee \mathsf{Q}) = \max(v(\mathsf{P}), v(\mathsf{Q})) \geq v(\mathsf{P}),$$

which would imply that Ivan cannot possibly be of higher rank than Jill.

Conditionals

The conventions employed by classical logic for handling negations, conjunctions, and disjunctions perfectly track the way these connectives are used in normal conversation. No training in logic is necessary to understand that an "and" statement is false if any of its atoms is false, and that an "or" statement is true if any of its atoms is true. (We should emphasize, however, that classical logic assumes we intend the "inclusive or" as opposed to the "exclusive or.")

By contrast, the proper treatment of conditional statements is a perennial problem for logicians. In mathematics we generally accept a "truth-functional" understanding of conditionals, which is to say that the truth of a conditional is determined completely by the truth of its component parts. This treatment of conditionals is useful for many purposes, but it plainly ignores some of the realities of regular conversation. For example, we normally expect the parts of a truthful conditional to have some relevance to one another. Keeping in mind that I am writing this in the United States, what are we to make of the statement, "If I am not in France, then I am not in Spain."? Classical logic says this is true, since the antecedent and conclusion are both true. This does not seem reasonable, however.

The account of conditionals used in classical logic suffers from other problems as well, in the form of arguments that are formally valid, but which can seem contrary to common sense in specific instances. The books by Priest [**8**] and Read [**9**] provide numerous examples.

Fuzzy logic inherits some of these problems, since it also treats conditionals as truth-functional. However, the particular function used to evaluate the truth value of a conditional has some interesting properties, as the three puzzles in this section will show.

Problem 7. *One day you meet Lana, who says, "If I am a knight, then I am a knave." What can you conclude about Lana?*

Solution: Let P denote the statement, "Lana is a knight." To simplify the notation, we shall define Q to be $\neg\mathsf{P}$. That is, Q is the statement, "Lana is a knave," Lana's statement is then equivalent to $\mathsf{P} \to \mathsf{Q}$.

Now, if Lana is fully knave or a quasiknave, then $v(\mathsf{P}) < v(\mathsf{Q})$. This implies that $v(\mathsf{P} \to \mathsf{Q}) = 1$, which is too high. If Lana is fully knight, then $v(\mathsf{P}) > v(\mathsf{Q})$. This implies that $v(\mathsf{P} \to \mathsf{Q}) < 1$, which is too low.

It follows that Lana is a quasiknight, but we can also conclude a bit more. Since $v(\mathsf{P}) > v(\mathsf{Q})$, we have that

$$v(\mathsf{P} \to \mathsf{Q}) = 1 - (v(\mathsf{P}) - v(\mathsf{Q})).$$

Moreover, we must also have $0.5 < v(\mathsf{P} \to \mathsf{Q}) < 1$. This is only possible if $v(\mathsf{P}) - v(\mathsf{Q}) < 0.5$, which in turn implies that $0.5 < v(\mathsf{P}) < 0.75$. Thus, Lana is a quasiknight who is between a quarter and halfway through her transition. □

Our next problem exhibits another interesting property of fuzzy conditionals. It will be convenient to refer to the "type" of an islander, by which we mean the categories fully knight, fully knave, quasiknight, and quasiknave.

Problem 8. *What can you conclude from the following dialog?*

Mark : *Nell is a knight. Also, Ozzy is a knave.*

Nell : *If I am a knave, then Ozzy is a quasiknight.*

Ozzy : *If I am a quasiknave, then Mark and Nell are of the same type.*

Solution: The most efficient solution to this problem involves the following observation: Suppose that P and Q are propositions with $v(\mathsf{Q}) = 0$. It then follows that $v(\mathsf{P}) \geq v(\mathsf{Q})$. We can now carry out the following calculation:

$$v(\mathsf{P} \to \mathsf{Q}) = 1 - (v(\mathsf{P}) - v(\mathsf{Q})) = 1 - v(\mathsf{P}) = v(\neg\mathsf{P}).$$

In the present case we define P to be the proposition, "Nell is a knave," and Q to be the proposition, "Ozzy is a quasiknight." Then Mark's first statement is $\neg\mathsf{P}$, while Nell's statement is equivalent to $\mathsf{P} \to \mathsf{Q}$. We will also define R to be the proposition, "Ozzy is a quasiknave."

We will now establish that Ozzy's statement has truth value 1, implying that he is fully knight. To do that, we note that if Ozzy really is a quasiknave, then we have that $v(\mathsf{R}) = 1$. This immediately implies that $v(\mathsf{Q}) = 0$. It now follows from our observation that since $v(\mathsf{Q}) = 0$, we have that $v(\mathsf{P} \to \mathsf{Q}) = v(\neg\mathsf{P})$, which implies that Mark and Nell are of the same type. We conclude that if Ozzy is a quasiknave then Mark and Nell are the same type, which is precisely what Ozzy asserted. It follows that Ozzy's statement has truth value 1, and therefore that Ozzy is fully knight.

This shows that Mark's second statement has truth value 0, implying that Mark is fully knave. Since we have already established that Mark and Nell are of the same type, we conclude that Nell is fully knave as well. So the solution is that Mark and Nell are fully knave, while Ozzy is fully knight. □

A further intriguing feature of fuzzy logic is that classical tautologies and classical contradictions must sometimes be reappraised. This is illustrated in the next problem. Let me remind you that a biconditional proposition, by which we mean a proposition employing the "if and only if" connective, is defined by the equivalence

$$\mathsf{P} \leftrightarrow \mathsf{Q} \equiv (\mathsf{P} \to \mathsf{Q}) \wedge (\mathsf{Q} \to \mathsf{P}).$$

It is a consequence of this definition that $v(\mathsf{P} \leftrightarrow \mathsf{Q}) = 1$ if and only if $v(\mathsf{P}) = v(\mathsf{Q})$.

Problem 9. *What can you conclude from the following dialog?*

Peta : *I am a knight or I am a knave.*

Also, Quin is a knave if and only if I am a quasiknight.

Quin : *I am a knight and I am a knave.*

Solution: On a classical island, only a knight can make Peta's first statement and only a knave can make Quin's statement. On fuzzy islands, however, there are other possibilities. It is always the case that one of the statements "I am a knight" and "I am a knave" must have a truth value that is strictly greater than 0.5, and one must have a truth value that is strictly smaller than 0.5, assuming the "I" refers to the same person in both statements. Since a conjunction is only as true as its least true conjunct, we see that Quin's statement has a truth value that is strictly smaller than 0.5. Therefore, she is either fully knave or a quasiknave. Likewise, since a disjunction is as true as its most true disjunct, we see that Peta's first statement has a truth value that is strictly greater than 0.5. We conclude that Peta is either fully knight or a quasiknight.

That leaves four possibilities for the types of Peta and Quin, but Peta's second statement allows us to narrow that down to one. Let us denote by P the proposition, "Quin is a knave," and by Q the proposition, "Peta is a quasiknight." Then Peta's second statement is equivalent to $\mathsf{P} \leftrightarrow \mathsf{Q}$.

Since Peta is either fully knight or a quasiknight, we know that

$$0.5 < v(\mathsf{P} \leftrightarrow \mathsf{Q}) \leq 1 \qquad \text{and} \qquad v(\mathsf{Q}) = 1 \text{ or } 0.$$

We also know that Quin is either fully knave or a quasiknave, which implies that $0.5 < v(\mathsf{P}) \leq 1$.

We next observe that

$$v(\mathsf{P} \leftrightarrow \mathsf{Q}) = v((\mathsf{P} \to \mathsf{Q}) \wedge (\mathsf{Q} \to \mathsf{P})) = \min(v(\mathsf{P} \to \mathsf{Q}), v(\mathsf{Q} \to \mathsf{P})).$$

If $v(\mathsf{Q}) = 0$, then Peta is not a quasiknight and must therefore be fully knight. But in this case we have

$$v(\mathsf{P} \to \mathsf{Q}) = 1 - (v(\mathsf{P}) - v(\mathsf{Q})) = 1 - v(\mathsf{P}) < 0.5,$$

which implies that $v(\mathsf{P} \leftrightarrow \mathsf{Q}) < 0.5$ as well. This is a contradiction. It follows that $v(\mathsf{Q}) = 1$, and Peta is actually a quasiknight.

From this we conclude that her second statement has a truth value that is strictly less than one. That is, $v(\mathsf{P} \leftrightarrow \mathsf{Q}) < 1$. But if $v(\mathsf{P}) = 1$, then we would have $v(\mathsf{P} \leftrightarrow \mathsf{Q}) = 1$, which is a contradiction. It follows that Quin is not fully knave.

Therefore, the solution is that Peta is a quasiknight and Quin is a quasiknave. □

Challenging *modus ponens*

It is not just tautologies and contradictions that must be reconsidered in the context of fuzzy logic. Even something as fundamental as *modus ponens*, by which we mean the argument form in which Q is said to follow from assuming the truth of P and $\mathsf{P} \to \mathsf{Q}$, must be reevaluated. That task will be undertaken in this section, but first we must ask a preliminary question. How should we understand "validity" in the context of fuzzy logic?

In classical logic, we say an argument is valid if the conclusion must be true whenever all of the premises are assumed to be true. In fuzzy logic, though, "truth" is a matter of degree. Informally, then, we might say that an argument in this context is valid if the conclusion must have a high truth value given that all of the premises have high truth values. This can be made more precise by deciding on a set of "distinguished" truth values, and then declaring an argument to be valid if the conclusion must have a distinguished truth value whenever all of the premises have distinguished truth values. For example, we might decide, arbitrarily, that a proposition P has a distinguished truth value if $0.8 \leq v(\mathsf{P}) \leq 1$.

This seems reasonable, but the next puzzle shows that it has some surprising consequences.

Problem 10. *You are observing a group of school kids. One of them, Rhea, approaches you and makes the following statements about some of her classmates:*

1. *Stan is a knight.*
2. *If Stan is a knight, then so is Theo.*
3. *If Theo is a knight, then so is Ursa.*

At this point, Ursa runs up to you and says, "Don't believe Rhea! She's a knave!" What can you conclude?

Solution: On a classical island, this conversation would be impossible. For suppose Rhea is fully knight. In this case, all three of her statements have truth value 1. A straightforward application of *modus ponens* would now show that Ursa must also be fully knight. In this scenario, however, Ursa's statement has truth value 0. This is a contradiction.

Now suppose Rhea is fully knave. In this case, her first statement has truth value 0. But then her second statement is a conditional with a false antecedent, which implies the whole statement is true. This, again, is a contradiction.

This would exhaust the possibilities on a classical island. On fuzzy islands, however, we have two additional possibilities. The first is that Rhea is a quasiknave. That this is impossible is shown by the following argument: If Rhea is a quasiknave, then her first statement has a truth value that is no greater than 0.5. But then her second statement is a conditional whose antecedent has a truth value no greater than 0.5. Now, an arbitrary conditional statement $\mathsf{P} \rightarrow \mathsf{Q}$ has truth value 1, when $v(\mathsf{P}) \leq v(\mathsf{Q})$, or truth value $1 - (v(\mathsf{P}) - v(\mathsf{Q}))$, when $v(\mathsf{P}) > v(\mathsf{Q})$. This implies that if $v(\mathsf{P}) < 0.5$, then $v(\mathsf{P} \rightarrow \mathsf{Q}) > 0.5$. This observation, applied to the specific case of Rhea's statements, implies that she has made a statement whose truth value is larger than 0.5. This is impossible if Rhea is a quasiknave.

Can Rhea be a quasiknight? This might seem impossible at first. Reasoning informally, we might say that since Rhea is a quasiknight, all of her statements have high truth values. An application of *modus ponens* would then suggest that the statement, "Ursa is a knight" must also have a high truth value. But this is not the case, since Ursa's statement has a low truth value in this scenario. (Keep in mind that as a quasiknight, Rhea is more knight than knave.)

This argument is a bit *too* informal, however. It is possible for Rhea to be a quasiknight, but only if we have something like the following scenario: We start by declaring that

P is the proposition: "Stan is a knight,"

Q is the proposition: "Theo is a knight,"

R is the proposition, "Ursa is a knight."

Rhea's statements are then, sequentially, P, $\mathsf{P} \to \mathsf{Q}$, and $\mathsf{Q} \to \mathsf{R}$. We must have that $v(\mathsf{R}) < 0.5$, since Ursa's statement has a low truth value when Rhea is a quasiknight. Now suppose we have something like $v(\mathsf{P}) = 0.7$, $v(\mathsf{Q}) = 0.4$, and $v(\mathsf{R}) = 0.1$. We would then compute:

$$v(\mathsf{P} \to \mathsf{Q}) = 1 - (0.7 - 0.4) = 0.7,$$
$$v(\mathsf{Q} \to \mathsf{R}) = 1 - (0.4 - 0.1) = 0.7.$$

These truth values are in the proper range for a quasiknight. We can conclude that Rhea is a quasiknight, from which it follows immediately that Ursa is a quasiknave.

However, the status of Stan and Theo is more ambiguous. In the scenario I just described, it is clear from the truth values of their statements that Stan is a quasiknight and Theo is a quasiknave. However, it is also viable to suppose that $v(\mathsf{P}) = 0.8$, $v(\mathsf{Q}) = 0.6$, and $v(\mathsf{R}) = 0.4$. In this case we would find that Stan and Theo are both quasiknights. On the other hand, it is viable to suppose that $v(\mathsf{P}) = 0.4$, $v(\mathsf{Q}) = 0.2$, and $v(\mathsf{R}) = 0.1$ In this case Stan and Theo would both be quasiknaves. □

This shows, surprisingly, that *modus ponens* is no longer a valid argument form in fuzzy logic. More precisely, we have shown that each link in a chain of implications can have a high truth value, while the conclusion of the final item has a low truth value.

Perhaps, though, this should be viewed as a feature, and not a bug, of fuzzy logic. Classical logic, you see, faces the problem of "sorites" paradoxes. The problem occurs whenever extreme ends of a continuum differ in some respect, but it is not possible to find a clear point on the continuum where the difference first manifests itself. The classic example involves the notion of a heap ("sorites" is the Greek word for "heap"). One grain of sand does not make a heap. Surely, though, adding one grain of sand to something that is not a heap cannot suddenly make it a heap. We conclude that two grains of sand is also not a heap. But now iterating the argument leads to the conclusion that there is no number of grains of sand that makes a heap, which is plainly absurd.

There has been some discussion in the professional logic literature as to whether sorites paradoxes constitute challenges to the universal applicability of *modus ponens* [**3, 4, 5**]. (It is worth noting, incidentally, that while *modus ponens* seems like the most obvious rule of deductive reasoning, it is not so clear that it applies universally, even without considering sorites paradoxes [**7**].)

The next puzzle shows how positing a continuum of truth values might evade the sorites problem. We shall assume that an islander's transition from fully knight to fully knave happens continuously, but very gradually. More precisely, we shall make two assumptions. The first is that the full transition occurs at a constant rate over a period of several days. The second is that for any time interval t, no matter how small, a person in the transitional phase is closer to the end of his transition after t has elapsed than he was before t had elapsed.

Problem 11. *This time Vito tells you about his friend Walt. He makes the following statements:*

1. *Walt is a knight.*
2. *If anyone is currently a knight, then he will still be a knight one second from now.*
3. *Therefore, Walt will still be a knight one second from now.*

What can you conclude about Vito and Walt? Is the argument valid?

Solution: Someone in the transitional phase will be very slightly less of a knight one second from now than he currently is. It follows that Vito's conditional statement has a conclusion whose truth value is very slightly smaller than the truth value of the antecedent. Thus, the whole conditional has a truth value that is very slightly less than one, which implies that Vito is a quasiknight. This now implies that Vito's first statement likewise has a truth value between 0.5 and 1, which implies that Walt is a quasiknight as well.

The argument is valid in the sense that whenever the premises have high truth values, the conclusion does as well. It might seem, therefore, that this is an instance of a sorites paradox, since iterating the argument leads to the conclusion that Walt will always be a knight, when we know that, in reality, after some number of seconds he will be fully knave. In this case, however, it is straightforward to resolve the paradox. Each time we iterate the argument, the conclusion is slightly less true than it was the previous time. Therefore, there is some specific number of iterations after which the conclusion will, for the first time, cease to have a distinguished truth value. After this number of iterations the argument will cease to be valid. □

The ability to defuse sorites paradoxes remains an important motivation for multi-valued and infinite-valued logics. By recognizing a continuum of truth values, we can claim that each link in the chain of implications in a sorites argument is less true than the one before, which implies that the chain is broken at some specific point.

It is debatable whether this approach is really successful [**8**], but we shall not pursue that question here.

The final problem

Let us close with a puzzle that requires a more detailed analysis than what has come before.

Problem 12. *What can you conclude from the following dialog?*

Xeno : *I am a knight and Yuki is a knave. Zack is a knave.*

Yuki : *If Xeno is a knight, then I am a knave.*

Zack : *Yuki is a quasiknight or Xeno is a knave.*

Solution: We shall have to be systematic in considering the various cases. Let us make the following definitions:

P is the proposition: "Xeno is a knight,"

Q is the proposition: "Yuki is a knave."

Then Xeno's first statement is equivalent to $\mathsf{P} \wedge \mathsf{Q}$. Yuki's statement is equivalent to $\mathsf{P} \rightarrow \mathsf{Q}$.

We can now distinguish two cases.

1. Suppose $v(\mathsf{Q}) \geq v(\mathsf{P})$.

   In this case, the conclusion of Yuki's conditional has a truth value that is not smaller than its antecedent. This implies that $v(\mathsf{P} \rightarrow \mathsf{Q}) = 1$. It follows that Yuki is fully knight. This tells us immediately that $v(\mathsf{Q}) = 0$. Since we are assuming $v(\mathsf{Q}) \geq v(\mathsf{P})$, we must also have that $v(\mathsf{P}) = 0$, implying that Xeno is fully knave.
2. Suppose $v(\mathsf{Q}) < v(\mathsf{P})$.

   Given this assumption, we immediately conclude that $v(\mathsf{Q}) < 1$. Since

   $$v(\mathsf{P} \wedge \mathsf{Q}) = \min(v(\mathsf{P}), v(\mathsf{Q})) = v(\mathsf{Q}) < 1,$$

   we know that Xeno is not fully knight. We now distinguish three further cases.

(a) Suppose that Xeno is a quasiknight. The analysis now proceeds in a manner similar to the first subcase. Since Xeno is assumed to be a quasiknight we have that $0.5 < v(\mathsf{P} \wedge \mathsf{Q}) < 1$. Also, we are still assuming that $v(\mathsf{Q}) < v(\mathsf{P})$. It then follows:

$$0.5 < \min(v(\mathsf{P}), v(\mathsf{Q})) < 1,$$

which implies

$$0.5 < v(\mathsf{Q}) < v(\mathsf{P}) < 1.$$

Consequently, $v(\mathsf{P}) - v(\mathsf{Q}) < 0.5$. Therefore, $0.5 < v(\mathsf{P} \rightarrow \mathsf{Q}) < 1$. This would again imply that Yuki is a quasiknight. So Xeno and Yuki are both quasiknights in this scenario.

(b) Suppose that Xeno is a quasiknave. Then

$$0 < v(\mathsf{P} \wedge \mathsf{Q}) < 0.5.$$

It follows that

$$0 < \min(v(\mathsf{P}), v(\mathsf{Q})) < 0.5.$$

Since we are assuming that $v(\mathsf{Q}) < v(\mathsf{P})$, we conclude that $0 < v(\mathsf{Q}) < 0.5$. But since Xeno is assumed to be a quasiknave, we also have that $0 < v(\mathsf{P}) < 0.5$. It follows that $0 < v(\mathsf{P}) - v(\mathsf{Q}) < 0.5$. Consequently, since we know that

$$v(\mathsf{P} \rightarrow \mathsf{Q}) = 1 - (v(\mathsf{P}) - v(\mathsf{Q})),$$

we have that $0.5 < v(\mathsf{P} \rightarrow \mathsf{Q}) < 1$. This implies that Yuki is a quasiknight. Thus, in this scenario we have that Xeno is a quasiknave and Yuki is a quasiknight.

(c) Finally, suppose that Xeno is fully knave. Then $v(\mathsf{P} \wedge \mathsf{Q}) = 0$. Also, $v(\mathsf{P}) = 0$. Since the first part of Yuki's conditional has truth value 0, we have that $v(\mathsf{P} \rightarrow \mathsf{Q}) = 1$. This implies that Yuki is fully knight.

Our analysis has revealed that there are two possible scenarios. One is that Xeno is fully knave and Yuki is fully knight. The other is that Yuki is a quasiknight and Xeno is a quasi-something. This implies that either Yuki is a quasiknight or Xeno is fully knave.

But this is precisely what Zack said! It follows that his statement has truth value 1, and therefore that he is fully knight. Thus, Xeno's second statement has truth value 0, implying that he is fully knave. This immediately implies that Yuki is fully knight.

So the solution is that Xeno is fully knave, while Yuki and Zack are both fully knight. □

There is certainly plenty of unexplored territory here for future work. I have spoken here as though "fuzzy logic" were a single thing. However, it is more correct to speak of different systems of fuzzy logic, just as there are different systems of geometry. How would our analysis of these puzzles have differed if we had chosen a different system? And what puzzle-making possibilities arise from other systems of nonclassical logic, such as modal and relevance logics?

It would seem there remain islands left to explore.

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Summary. Puzzles about knights, who only make true statements, and knaves, who only make false statements, have long been a mainstay of classical logic. They are valuable not just as recreational puzzles, but as a pedagogical device for exploring fundamental issues in logic. However, the possibilities for puzzles based on nonclassical logics have been mostly unexplored. In this paper we consider knight/knave dialogs based on fuzzy logic, in which a truth value can be any real number between zero and one inclusive. Following the example of classical logic, our purpose is both recreational and educational.

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Limit Comparison for Series

For this question I must disclaim,
One must be on top of their game,
Limit a over b,
It's 1 you can see,
Meaning a and b are the same.

— Clayton Arundel, undergraduate student, Gordon College
Communicated by Karl-Dieter Crisman, Gordon College

Proof Without Words: Alternating Row Sums in Pascal's Triangle

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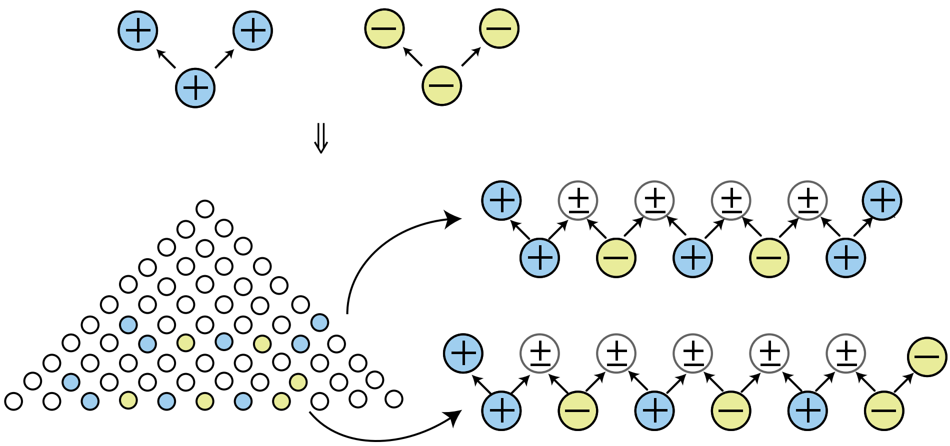
Theorem. For any integers $0 \le j \le m \le n$,

$$\sum_{k=j}^{m} (-1)^k \binom{n}{k} = (-1)^j \binom{n-1}{j-1} + (-1)^m \binom{n-1}{m},$$

and in particular if $j = 0$ and $m = n$, then $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$ (by defining as usual $\binom{n-1}{-1} = 0 = \binom{n-1}{n}$).

Proof. For simplicity we show the case when j is even; the odd cases can be obtained by reversing the role of $+$ and $-$.



$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \Rightarrow \sum_{k=j}^{m} (-1)^k \binom{n}{k} = (-1)^j \binom{n-1}{j-1} + (-1)^m \binom{n-1}{m}.$$

Summary. Based on the Pascal's identity, we visually demonstrate that the alternating sum of consecutive binomial coefficients in a row of Pascal's triangle is determined by two binomial coefficients from the previous row.

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The Limiting Value of a Series With Exponential Terms

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There are many intriguing problems in the theory of sequences and series that require detailed arguments to establish convergence. Among these are the class of problems involving the determination of the limiting value of a sequence defined by

$$b_n = \sum_{k=1}^{n} a_{k,n}. \tag{1}$$

Often times a closed form of $\lim\limits_{n\to\infty} b_n$ may be difficult to obtain because of the manner in which the variables k and n are coupled in the $a_{k,n}$ terms. In some cases the $\lim\limits_{n\to\infty} b_n$ can be obtained using Riemann sums as in the following example:

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k^2}} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\sqrt{1+(k/n)^2}} = \int_0^1 \frac{1}{\sqrt{1+x^2}}dx = \ln\left(1+\sqrt{2}\right).$$

For other problems, inequalities and the squeeze theorem prove useful. For example,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k}} = 1$$

can be proved by appealing to the fact that

$$\frac{1}{\sqrt{n^2+n}} \leqslant \frac{1}{\sqrt{n^2+k}} \leqslant \frac{1}{\sqrt{n^2+1}},$$

for positive integers k between 1 and n. However, many problems in the form of (1) will not yield immediate solutions by the aforementioned methods. A case in point is the sequence defined by $s_n = \sum_{k=1}^{n}(k/n)^n$. This intriguing sequence is mentioned in an exercise in Knopp's famous book on infinite series [**1**], and it has also been the subject of some recent articles in THIS MAGAZINE where the sequence s_n has been shown to converge to $e/(e-1)$. Specifically, in [**2**] Spivey appeals to the Euler–Maclaurin summation formula to prove this limit while Holland [**3**] presents an "elementary but *ad-hoc*" proof along with a more sophisticated proof that utilizes the Lebesgue dominated convergence theorem. Holland also notes that the limit of s_n can be obtained with Tannery's theorem [**4**], and in an earlier article in THIS MAGAZINE, Boas [**5**] makes the same observation.

There are several other series that look similar to s_n but are more manageable as far as determination of convergence is concerned. For example, it is straightforward to show that as n approaches infinity, the expression $\sum_{k=1}^{n}(1/n)^k$ converges to 0, and similarly $\sum_{k=1}^{n}(1/k)^n$ converges to 1. Another well known variation is $\sum_{k=1}^{n}(1/k)^k$,

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which has a limiting value of $\int_0^1 (1/x)^x\,dx \approx 1.291286$, a value sometimes referred to as the "sophomore's dream constant" [**6**].

The variation that this paper focuses on is

$$t_n = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^k.$$

Interestingly enough, it turns out that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = e/(e-1)$, despite the fact that all fractional terms of t_n are larger than corresponding terms in s_n. In the following discussion, three proofs of this fact are presented in order of increasing sophistication. The first proof evaluates $\lim_{n\to\infty} t_n$ by means of a comparison with the sequence s_n. The second uses techniques that are usually covered in a first course in real analysis. In the final proof Tannery's theorem is stated and applied to efficiently demonstrate $\lim_{n\to\infty} t_n = e/(e-1)$.

A proof by comparison

For each positive integer n let $g_n(x) = (x/n)^x$. The function g_n is convex on the interval $(1, n)$ and has a local minimum at n/e. For $n > 8$,

$$g_n(n/2) = (1/2)^{(n/2)} < 4/n^2 = g_n(2),$$

so that $4/n^2$ is the largest value of g_n on the interval $(2, n/2)$. Consequently, for $n > 8$

$$\sum_{k=1}^{\lfloor n/2\rfloor} (k/n)^k < \frac{1}{n} + \frac{4}{n^2}(n/2 - 1) < \frac{3}{n},$$

and thus

$$\lim_{n\to\infty} \sum_{k=1}^{\lfloor n/2\rfloor} (k/n)^k = 0. \tag{2}$$

Next define

$$f_n(x) = \left(\frac{x}{n}\right)^x - \left(\frac{x}{n}\right)^n.$$

We will show $\lim_{n\to\infty} \sum_{k=1}^{n} f_n(k) = 0$, which is equivalent to

$$\lim_{n\to\infty} (t_n - s_n) = 0.$$

Now $0 \leqslant f_n(x) \leqslant (x/n)^x$ for x on $[1, n]$. Then from (2) and the squeeze theorem for sequences we have

$$\lim_{n\to\infty} \sum_{k=1}^{\lfloor n/2\rfloor} f_n(k) = 0.$$

Therefore, all that remains is to show

$$\lim_{n\to\infty} \sum_{k=\lfloor n/2\rfloor+1}^{n} f_n(k) = 0. \tag{3}$$

Before considering this summation, it will be helpful to first identify any extrema of the function f_n. Differentiating f_n yields

$$f_n'(x) = \left(\frac{x}{n}\right)^x \left[1 + \ln\left(\frac{x}{n}\right)\right] - \left(\frac{x}{n}\right)^{n-1}.$$

After setting the derivative to zero and using the substitution $u = x/n$, the quantities n and u can be separated as follows:

$$n = \frac{\ln(1 + \ln(u)) + \ln(u)}{(1-u)\ln(u)}. \tag{4}$$

The values of u in (4) correspond to critical numbers of f_n, and n can be viewed as a function of u over the interval $0 < u < 1$. The function $n(u)$ is decreasing on $(0, 0.4825)$, increasing on $(0.4825, 1)$, and has a minimum value at $(u, n) \approx (0.4825, 5.3921)$. Therefore, for any fixed $n \geqslant 6$, there are two values of u which satisfy (4), and thus the function $f_n(x)$ has two critical numbers on $(1, n)$. We will denote these by α_n and β_n, and observe that these satisfy the inequality $\alpha_n < 0.4825n < \beta_n$. An analysis of the sign of $f_n'(x)$ reveals that the function f_n has a local minimum at α_n and a local maximum at β_n. It can be shown that $f_n'(n-3) < 0$ and $f_n'(n-2) > 0$ for $n \geqslant 6$, thus the critical number β_n lies between $(n-3)$ and $(n-2)$. (For $n \leqslant 5$ the function $f_n(x)$ is strictly decreasing on $(0, 1)$.)

The summation in (3) can be viewed as a Riemann sum of $f_n(x)$ over $[n/2, n]$ with subinterval width of 1. Since the local maximum of $f_n(x)$ occurs between $n-3$ and $n-2$, and because $f_n(n-2) > f_n(n-3)$, a lower Riemann sum $f_n(x)$ over $[n/2, n]$ would exclude $f_n(n-2)$. (See Figure 1.) Then a proper upper bound for the summation in (3) is the sum of a definite integral and $f_n(n-2)$. Specifically,

$$\sum_{k=n/2+1}^{n} f_n(k) < \int_{n/2}^{n} f_n(x)\,dx + f_n(n-2). \tag{5}$$

Clearly, $\lim_{n\to\infty} f_n(n-2) = 0$. Now all that remains is to show that the integral in (5) also approaches 0 in this same limit.

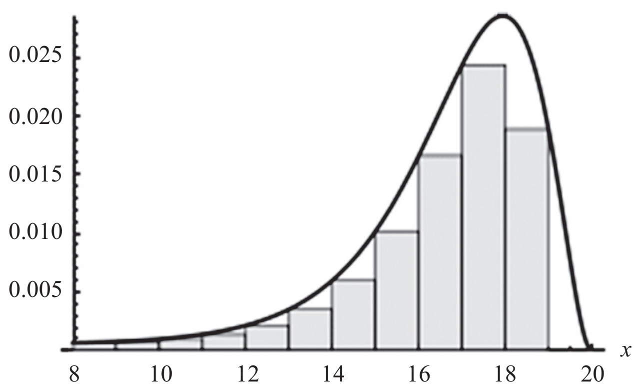

Figure 1 Lower Riemann sum for $f_{20}(x)$ for $8 \leqslant x \leqslant 20$. The area of rectangles shown here include the terms of the sum $\sum f_n(k)$ for $k = 20, 19, 17, 16, \ldots, 9, 8$. Note that the term corresponding to $k = 18 = n - 2$ is not included.

With a change of variables $x = nu$, the integral in (5) is

$$\int_{\frac{n}{2}}^{n} f_n(x)\,dx = n\int_{\frac{1}{2}}^{1} u^{nu}du - \frac{n}{n+1}\left(1 - \frac{1}{2^{n+1}}\right). \tag{6}$$

Let I_n denote the integral portion on the right-hand side of (6), and rewrite this as

$$I_n = n\int_{\frac{1}{2}}^{1} e^{nu\ln(u)}du = \int_{\frac{1}{2}}^{1} \frac{e^{nu\ln(u)}}{(1+\ln(u))} n\,(1+\ln(u))\,du.$$

Next, integrate by parts to obtain

$$I_n = 1 - \frac{(1/2)^{(n/2)}}{1-\ln(2)} + \int_{\frac{1}{2}}^{1} \frac{u^{nu}}{u\,(1+\ln(u))^2}du. \tag{7}$$

Let J_n denote the integral in Eq. (7). Since $1 \leqslant 1/u \leqslant 2$ in the integrand, the following inequality holds:

$$J_n \leqslant \frac{2}{n}\int_{\frac{1}{2}}^{1} \frac{nu^{nu}\,(1+\ln(u))}{(1+\ln(u))^3}du.$$

Apply another integration by parts, and use the fact that $0 < u^{nu} \leqslant 1$ to obtain

$$J_n \leqslant \frac{2}{n}\left(1 + 3\int_{\frac{1}{2}}^{1} \frac{1}{u\,(1+\ln(u))^4}du\right).$$

Since the integrand in the above equation does not exceed 2, we obtain $J_n \leqslant 8/n$. Thus by (7) we have $\lim_{n\to\infty} I_n = 1$, and because the expression after the integral on the right side of Eq. (6) approaches -1 as n approaches infinity,

$$\lim_{n\to\infty}\int_{\frac{n}{2}}^{n} f_n(x)\,dx = 0.$$

Therefore, $\lim_{n\to\infty}(s_n - t_n) = 0$, which implies $\lim_{n\to\infty} t_n = e/(e-1)$.

A direct proof

A direct approach to proving $\lim_{n\to\infty} t_n = e/(e-1) = S$ can be accomplished as follows. First let

$$t_n = t_{1,n} + t_{2,n} + t_{3,n},$$

where the parts corresponds to the following summations (respectively):

$$t_n = \sum_{k=1}^{\lfloor n/2\rfloor}(k/n)^k + \sum_{l=m}^{n-1-\lfloor n/2\rfloor}\left(1-\frac{l}{n}\right)^{n-l} + \sum_{l=0}^{m-1}\left(1-\frac{l}{n}\right)^{n-l}. \tag{8}$$

In (8) the substitution $l = n - k$ was used to create $t_{2,n}$ and $t_{3,n}$, and m is an integer less than $n/2$ which will be specified later. Now for a fixed value of l, we know that

$$\lim_{n\to\infty}\left(1-\frac{l}{n}\right)^{n-l} = e^{-l}.$$

Also,

$$\sum_{l=0}^{m-1} e^{-l} = \frac{e^m - 1}{e^m - e^{m-1}} = S_m.$$

Since

$$e^{-l/2} > \left(1 - \frac{l}{n}\right)^{n/2} > \left(1 - \frac{l}{n}\right)^{n-l},$$

for $l = 1, 2, \ldots, \lfloor n/2 \rfloor$, using geometric series arguments

$$t_{2,n} < \left[\left(\frac{1}{\sqrt{e}}\right)^m - \left(\frac{1}{\sqrt{e}}\right)^{n/2}\right]\left(\frac{\sqrt{e}}{\sqrt{e}-1}\right) < 3\left(\frac{1}{\sqrt{e}}\right)^m. \tag{9}$$

With these statements in mind, we can proceed with the proof. Let $\varepsilon > 0$, and select m so that $|S_m - S| < \varepsilon/4$ and $3e^{-m/2} < \varepsilon/4$, where the latter inequality is motivated by (9). Then, for any $n > 2m + 1$, we have $m \leqslant n - 1 - n/2$, where $t_{2,n} < \varepsilon/4$ and so, for such n,

$$|t_n - S| \leqslant t_{1,n} + \varepsilon/4 + \left|t_{3,n} - S_m\right| + \varepsilon/4 = t_{1,n} + \left|t_{3,n} - S_m\right| + \varepsilon/2. \tag{10}$$

Since m is now fixed, the sum rule for limits tells us that $\lim_{n\to\infty} t_{3,n} = S_m$, and from (2) we know that the sequence $t_{1,n}$ converges. Hence, for sufficiently large n both $\left|t_{3,n} - S_m\right|$ and $t_{1,n}$ will be less than $\varepsilon/4$, and therefore by (10),

$$\limsup_{n\to\infty} |t_n - S| < \varepsilon.$$

And since ε was arbitrary, $\lim_{n\to\infty} t_n = S = e/(e-1)$.

Proof by Tannery's theorem

The most straightforward way to prove this limit is by making use of Tannery's theorem [**4**]. This theorem says that if $\lim_{n\to\infty} v_k(n) = L_k$ for each k, and if $|v_k(n)| \leqslant M_k$, where $\sum M_k$ is convergent, then

$$\lim_{n\to\infty} \sum_{k=1}^{p(n)} v_k(n) = \sum_{k=1}^{\infty} L_k,$$

where $p(n)$ approaches infinity as n approaches infinity. It is important to note that the terms M_k are independent of n. Bromwich [**4**] notes that Tannery's theorem and the proof are substantially the same as the M-test due to Weierstrass [**7**], which is used to determine uniform convergence of a series of functions over a given domain. Tannery's theorem is, in essence, a discrete version of the Lebesgue's dominated convergence theorem. To apply this theorem, we express t_n as

$$t_n = \frac{1}{n} + \sum_{k=0}^{\infty} v_k(n),$$

where

$$v_k(n) = \begin{cases} (1 - k/n)^{n-k}, & \textit{if } 0 \leqslant k \leqslant n-2, \\ 0, & \textit{if } k \geqslant n-1. \end{cases}$$

Now $\lim_{n\to\infty} v_k(n) = e^{-k}$, and for any fixed k satisfying $0 \leqslant k \leqslant n-2$, the function $v_k(n)$ decreases with increasing n, and therefore has its maximum value at $n = k+2$. Thus,

$$v_k(n) \leqslant \frac{4}{(k+2)^2} = M_k.$$

Since $\sum M_k$ is convergent, all conditions of Tannery's theorem are satisfied, and

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \left(\sum_{k=0}^{n-2} e^{-k} \right) = \frac{e}{e-1}.$$

This third approach is clearly the most elegant and concise way to evaluate our limit, and it demonstrates Tannery's theorem as a powerful tool that can be used to establish results which are not easily obtained with other well-known techniques.

Further generalizations

Many of the proof techniques discussed can be applied to more general summations. For example, if z is a complex number, α and β are positive real numbers, then

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{z+k}{n} \right)^{\alpha n+\beta} = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{z+k}{n} \right)^{\alpha k+\beta} = \frac{e^{\alpha+\alpha z}}{e^{\alpha}-1}.$$

The interested reader may want to work through the details of a formal proof of this statement, as well as consult the references to find other techniques for determining convergence of sequences that are defined by series.

Acknowledgments The author would like to thank the reviewers for alerting me to the proof involving Tannery's theorem and for many other helpful suggestions.

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Summary. A sequence defined by a series of exponential terms was discussed by Spivey [*Math. Mag.* **79** (2006) 61–65] and Holland [*Math. Mag.* **83** (2010) 51–54]. In this paper, a simple variation of this sequence is discussed, and three proofs are given to establish that both sequences have the same limiting value. A proof by comparison uses an asymptotic approximation of an improper integral that models the difference between the sequences. Then a direct proof of this fact is obtained using well known inequalities involving exponential functions along with basic properties of geometric series. Finally, Tannery's theorem is defined and applied to efficiently demonstrate that the sequence converges to the ratio of e and $(e-1)$.

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ACROSS

1. ___ MAG, i.e., this journal
5. Oracles
10. Singer Domino
14. Colonel Mustard's game
15. Apportion
16. "I cannot tell ___"
17. Bonus or more, in ads
18. Originator of Zeno's Achilles and tortoise story
19. ___-o-meal
20. ___ winner (candidate who defeats all others head-to-head)
22. ___ count (procedure in which voters assign $n, n-1, \ldots, 1$ points to n candidates)
23. *True* ___
24. Astuteness
25. "And all the ships ___"
28. Sequences and ___
31. Generate
33. Grand Canyon feature
34. η
37. Washington, D.C. has 435
41. hard, harder, hard___
42. ___ Lanka
43. Plácido Domingo, José Carreras, and Luciano Pavarotti
44. Stock portions
47. A ___ in one's side
48. David Bowie's "Space ___"
51. "Flashdance" singer
53. Author with Fishburn of *Approval Voting*
54. First past the post, also known as ___
59. Initialism for a 50+ organization
60. "___ is human"
61. What you should do on November 8, 2016
62. 1-Down plus CMLX
63. Applause accompaniment
64. English public school
65. Start of sowing adage
66. Scatter
67. Drains one's energy

DOWN

1. 1190, in Ancient Rome
2. Sax type
3. ___ the page
4. Dreaded apparatus for those with braces
5. Author of *The Geometry of Voting*
6. Chooses a candidate
7. If, then, ___
8. $\frac{-b+\sqrt{b^2-4ac}}{2a}$ is one for $ax^2+bx+c=0$
9. Brand of motor oil additive
10. Cameron Crowe's *Almost* ___
11. Five-___ chili
12. $\sim$
13. ___ example (serve as a role model)
21. Declaim
22. Pre-AD
24. Set sights, as to shoot
25. 4840 square yards
26. Uno, dos, ___
27. Aug., ___, Oct.
29. Bert and ___
30. Dye pack brand
32. English class homework
34. What Rachael Ray cooks with, abbr.
35. Area under US jurisdiction, abbr.
36. Org. type, such as the publisher of this MAG
38. Msg. when dividing by zero
39. Prefix for a four-sided polyhedron
40. How soccer and college basketball games are played
44. Type of group
45. Bkln ___
46. Graph of logistic function, loosely
48. 44th President of the US
49. 1/8-ounce measures
50. Pride : Elizabeth as Prejudice : ___
52. Nobel Prize economist known for his impossibility theorem
54. Left, on a ship
55. Shakespeare's King
56. ι
57. Roof of Bandit's Trans Am
58. Feelings of longing, as for ice cream
60. Atlanta-based cable network, abbr.

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Mathematics and Elections

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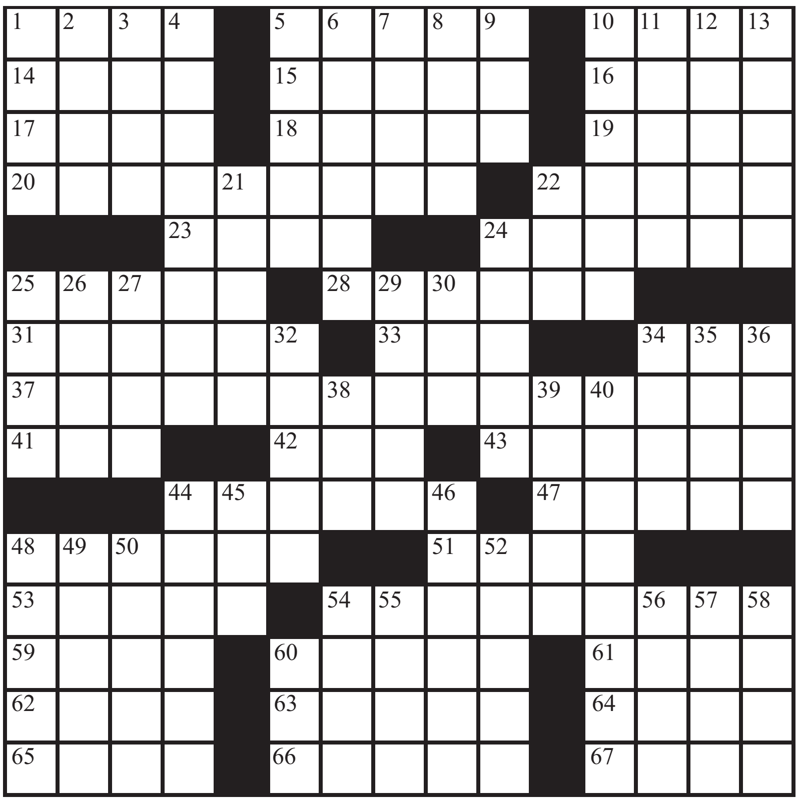

Clues start at left, on page 288. The Solution is on page 315.

Extra copies of the puzzle can be found at the MAGAZINE's website, www.maa.org/mathmag/supplements.

Crossword Puzzle Creators

If you are interested in submitting a mathematically themed crossword puzzle for possible inclusion in MATHEMATICS MAGAZINE, please contact the editor at mathmag@maa.org.

Dick Termes: Art of the Sphere*

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Dick Termes is an artist, author, teacher, and owner of the Termesphere Gallery, which is housed in a geodesic dome near Spearfish, South Dakota. In 1968, he developed an innovative process that produces a full panoramic perspective painting using the sphere as a canvas; the result he calls a Termesphere. David met with Dick at his gallery in August 2015 to discuss his background and art. A portion of this interview appears below. Accompanying artwork appears on the following pages: 250 and 266.

Q: *How long have you been creating art?*

DT: I started in high school when I was a junior; that was in 1958. I explored probably every type of art, and I'm still fascinated with all the different kinds of things that can come out of your head.

Q: *When did you develop the idea of a Termesphere?*

DT: I started in '68–'69. I was at the University of Wyoming working on my master's degree when I hit on the six-point perspective on the sphere idea. I went to school for two more years for an MFA at the Otis Art Institute in Los Angeles. I pretty much got to work just on the sphere for my MFA thesis. They just left me alone, and I got my degree out of it, which is cool. My instructors had no clue what I was doing and they would come in and they would just have coffee with me and we'd talk.

Q: *What's the largest sphere that you've created?*

DT: Seven-and-a-half feet. It is in Douglas, Wyoming, at the Law Enforcement Academy.

Q: *Can you describe the six-point perspective concept you use in a Termesphere?*

The six-point perspective is the idea that if you're inside a cubicle world and you had a transparent ball on your head, then there would be six equal distant points on the transparent ball and any cubicle world will project to those six points. You can be anywhere within that cubicle world, like a cathedral. You could be way up in the corner; however, the six points never move, but the world adjusts. That's the math side of the sphere.

On the sphere, there's actually three sets of poles; if you think of yourself in a cubicle room, the set of lines that go north also go to the south; all the lines that project off to the east also go to the west. All the lines and all the columns that are running up and down project to a point above your head and project to a point below your feet. There's those six points but three sets of poles that everything projects to.

Q: *When you start to create a Termesphere, do you start by drawing scaffolding lines?*

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MSC: Primary 01A70; Amy L. Reimann (MR Author ID: 1118776) and David A. Reimann (MR Author ID: 912704)

Figure 1 Termespshere Gallery. The gallery is housed inside a geodesic dome and contains many Termespheres suspended from slowly rotating motors.

DT: Sometimes I just deal with geometry grids that are substructures. If I just find the six points, then I don't need the grid anymore. I know what I'm doing on it. In the early years I would grid it out so I would really feel safe about what my projections were, but now I use a flexible tape whatever the length of halfway around the ball is. I tape it to the points, then I draw the line so it's done in a very precise way.

Q: *Do you do a lot of flat two-dimensional works as well?*

DT: No, none at all. I probably have messed with a couple in all of these years, and that was mostly right at the beginning of my career. I really wasn't planning to focus on the sphere, but it sort of grabbed me. If you think about it, the flat world of painting, Euclidean geometry has a lot of influence on what you can do. But we've spent 2,000 years on it. With the flat world, that's basically what we've done. If you think about the spherical geometries, it's a whole different set of problems . . . different ways, patterns fit on the sphere. No one had played in that arena. So I was given this opportunity of using this new canvas. It allows me to do different things and allows me to capture complete environments too—that's the six-point perspective, and all the new geometries that fit on a sphere give me a whole different set of concepts.

Q: *Who are some of your inspirations?*

DT: Buckminster Fuller was one of my heroes. He's a great inspiration. He gave a lecture at Black Hills State University, and I got to pick him up at the airport and spend a couple days with him and his wife. And so that's where [my studio and gallery] domes came from. When he left I thought, "Oh God, I've got to do one of these." I already had been painting on spheres and as you can see, they fit in the domed gallery very well. I'm also inspired by artists that really took chances like Picasso, Paul Klee, and Seurat. I enjoy people that took off on their own and made the art world come to them, which is certainly what's happening here too. I actually connect easier with mathematicians than I do with the art world, which was Escher's problem too but the arts are coming around.

Q: *You said M. C. Escher was an inspiration for you as well.*

DT: Yes, Escher, for sure. I had done a couple years of spheres when people kept saying "you really look like Escher," and I thought "oh, I better study this guy a little bit." And then as soon as I started studying him, I realized that my thinking really was

Figure 2 Dick Termes in his studio with *Indra's Net* (2015), a 24-inch diameter Termesphere. Clear portholes allow the interior to be illuminated to allow the viewer to see the interior scene, which is painted differently than the exterior.

very connected. Every time I'd come up with an idea, he had played with it already on the flat dimension, but I was lucky that I could move it into the three-dimensional world.

Q: *Are you still creating a lot of works every year?*

DT: Yes, I don't think I've slowed down. Maybe I'm more precise about what I'm doing than ever before.

Q: *What are you currently working on?*

DT: A commissioned piece I want to show you is for a psychologist on the West Coast. We talked for three months, off and on, to try and get something zeroed in to what she thought of and what would fit with what I do. It's a big transparent sphere, where you have holes to look in and see what's on the inside (see Figure 2). The inside is all Indra's Net, a Hindu philosophy about what holds the whole universe together. I relate to it scientifically because many scientists spend a lot of the last of their life trying to find the theory of everything, and they're looking for that Indra's Net. What is it that holds all of this together? What are the basic elements? That's interesting to me and, just like scientists, I enjoy that. You're given this mind, you're trying to explore the world, and we should try and figure it out. It's a fascinating place. That piece is turning out quite well. It has a lot of realism in it too but moves from insideness to outsideness in strange ways.

Q: *If someone is interested in seeing your art, your studio here in Spearfish is a good place to see that. Are there other good places to see your art?*

DT: Well, the Internet is a good place to start, being as I live out here in the boonies. I probably get more people visiting Termespheres.com or viewing Termespheres on YouTube than actually visiting my studio. But it's amazing how many people will take the effort to make it out here.

PROBLEMS

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Correction

In Problem 1995 of the April issue, the matrix M should have $m_{i,j} = a_i/s_j$ for all $i \neq j$. Thanks to Robert Calcaterra from the University of Wisconsin-Platteville for catching the error.

Proposals

To be considered for publication, solutions should be received by March 1, 2017.

2001. *Proposed by Herb Bailey and Dianne Evans, Rose-Hulman Institute of Technology, IN.*

Fix positive integers n and k. Numbers are drawn one at a time with replacement from an urn containing one of each of the first n positive integers. Find the expected number of drawings needed until k successive drawings are all ones.

2002. *Proposed by Constantin P. Niculescu, Craiova, Romania and Gabriel T. Prăjitură, SUNY Brockport, NY.*

Call a real sequence $\{x_n\}_{n\in\mathbb{N}_+}$ *dense in average* if $\left\{x_1, \frac{x_1+x_2}{2}, \frac{x_1+x_2+x_3}{3}, \ldots\right\}$ is dense in $\mathbb{R}$.

(a) Show that there are sequences that are dense in $\mathbb{R}$ but not dense in average.

(b) Prove that every sequence that is dense in $\mathbb{R}$ has a subsequence that is both dense and dense in average.

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We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals must always be accompanied by a solution and any relevant bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Proposals and solutions should be written in a style appropriate for this Magazine.

Effective immediately, authors of proposals and solutions should send their contributions using the Magazine's submissions system hosted at `http://mathematicsmagazine.submittable.com`. *More detailed instructions are available there. We hope that this online system will help streamline our editorial team's workflow while still proving accessible and convenient to longtime readers and contributors. We encourage submissions in PDF format, ideally accompanied by LaTeX source. General inquiries to the editors should be sent to* `mathmagproblems@maa.org`.

2003. *Proposed by Julien Sorel, Columbus, GA.*

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(0) = 0$, and

$$f(x) = \int_0^x \cos\frac{1}{t}\,\cos\frac{3}{t}\,\cos\frac{5}{t}\,\cos\frac{7}{t}\,dt$$

for $x \neq 0$. Show that f is differentiable and $f'(0) = 1/8$.

2004. *Proposed by Mihàly Bencze, Brasov, Romania.*

For every positive integer n, prove the identity

$$\sum_{k=1}^{n-1} \binom{n}{k} \frac{kn^{n-k}}{k+1} = \frac{n(n^n - 1)}{n+1}.$$

2005. *Proposed by Daniel Ullman, George Washington University, Washington, DC.*

Let $S = \{3 \cdot 2^k - 2 : k \in \mathbb{N}\} = \{1, 4, 10, \dots\}$. For every nonempty finite subset $A \subset \mathbb{N}$, let $\pi(A) = \prod_{k \in A} k$. Compute

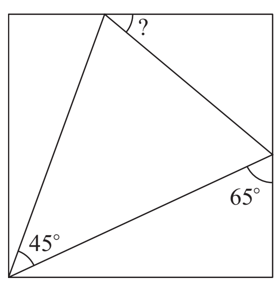
$$\sum \frac{1}{\pi(A)},$$

where the sum is taken over all finite nonempty subsets A of S.

Quickies

1063. *Proposed by Poo-Sung Park, Kyungnam University, Changwon, Korea.*

A triangle is inscribed in a square as shown in the picture. What is the measure of the angle with the question mark?



1064. *Proposed by José M. Pacheco and Ángel Plaza, Spain.*

For every positive integer n, show that

$$\int_0^1 \frac{dx}{x^{n-1} + \cdots + x + 1} \geq \frac{1}{1 + \frac{1}{2} + \cdots + \frac{1}{n}}.$$

Solutions

A Diophantine equation of degree 8 June 2015

1971. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Find all pairs of integers (x, y) such that

$$x^8 + (y^2 + y - 1)(4 - 3x^4) = 2.$$

Solution by Sungjun Choi, Institute of Science Education for the Gifted and Talented, Yonsei University, Seoul, Korea.
The equation is equivalent to

$$y^2 + y - 1 = \frac{2 - x^8}{4 - 3x^4},$$

since $4 - 3x^4 \neq 0$ for every integer x. Multiplying both sides by 9, we have

$$9(y^2 + y - 1) = 9 \cdot \frac{2 - x^8}{4 - 3x^4} = 4 + 3x^4 + \frac{2}{4 - 3x^4}.$$

If integers x and y satisfy the last equation, then $4 - 3x^4$ is a divisor of 2, so

$$4 - 3x^4 = 1, -1, 2 \text{ or } -2 \quad \Rightarrow \quad 3x^4 = 2, 3, 5 \text{ or } 6.$$

However, since x is an integer, we must have $3x^4 = 3$, hence $x = 1$ or $x = -1$. For either of these values of x, the original equation gives $y^2 + y - 2 = 0$, so $y = 1$ or $y = -2$. Hence, all possible solutions are

$$(x, y) \in \{(1, 1),\ (1, -2),\ (-1, 1),\ (-1, -2)\},$$

and all of them satisfy the original equation.

Also solved by Robert A. Agnew, Adnan Ali (India), Jeremiah Bartz, Michel, Bataille (France), Brian D. Beasley, John Christopher, Dalton Cowan, Tim Cross (UK), Joseph DiMuro, Miles Johnson, John Ferdinands, Dmitry Fleischman, John Gately, José Hernández (Mexico), Tom Jager, Graham Lord, Missouri State University Problem Solving Group, Titu Zvonaru & Neculai Stanciu (Romania), Taelor Randa, Herman Roelants (Belgium), Mehtaab Sawhney, Nicholas C. Singer, Digby Smith (Canada), David Stone & John Hawkins, Skidmore College Problem Group, Jan Verster (Canada), Joseph Walegir, Haohao Wang & Jerzy Wojdylo, Edward White, Lienhard Wimmer, Yuanyuan Zhao and the proposer. There were 2 incomplete or incorrect solutions.

An integral equation involving a geometric sum June 2015

1972. *Proposed by Marcel Chirita, Bucharest, Romania.*

Let $n \geq 2$ be an integer. Determine all continuous functions $f : [1, \infty) \to \mathbb{R}$ such that

$$\int_x^{x^n} f(t)dt = \int_1^x (t^{n-1} + t^{n-2} + \cdots + t) f(t)dt$$

for every $x \in [1, \infty)$.

Solution by Michel Bataille, Rouen, France.
We show that the functional equation is solved precisely by all functions of the form

$$f(x) = \begin{cases} C & (x = 1), \\ \dfrac{C}{x} \cdot \dfrac{x-1}{\ln x} & (x > 1), \end{cases} \tag{1}$$

where C is an arbitrary real constant.

First we prove that all solutions are of the form (1). Suppose that f is a solution. Add $\int_1^x f(t)dt$ to both sides of the given functional equation to obtain

$$\int_1^{x^n} f(t)dt = \int_1^x (1 + t + \cdots + t^{n-1}) f(t)dt. \tag{2}$$

Since f is continuous, differentiation of both sides of (2) with respect to x gives

$$nx^{n-1} f(x^n) = (1 + x + \cdots + x^{n-1}) f(x), \tag{3}$$

by the fundamental theorem of calculus. Let $g(x) = xf(x)$. Since f is continuous on $[1, \infty)$, so is g. From (3) and the geometric formula we obtain the functional equation

$$g(x^n) = \frac{1}{n} \cdot \frac{x^n - 1}{x - 1} g(x) \quad \text{for } x > 1.$$

Since $x \mapsto x^{1/n}$ maps $(1, \infty)$ into itself, we may substitute $x^{1/n}$ for x above to get

$$\frac{g(x)}{x-1} = \frac{1}{n} \cdot \frac{g(x^{1/n})}{x^{1/n} - 1} \quad \text{for } x > 1.$$

From this equation and induction we have

$$\frac{g(x)}{x-1} = \frac{1}{n^k} \cdot \frac{g(x^{1/n^k})}{x^{1/n^k} - 1} \quad \text{for } x > 1 \text{ and every positive integer } k. \tag{4}$$

For fixed $x > 1$, as $N \to \infty$, we have $x^{1/N} \to 1$, $g(x^{1/N}) \to g(1)$, and $N(x^{1/N} - 1) \to \ln x$ since $x^{1/N} = \exp((\ln x)/N) = 1 + (\ln x)/N + \mathcal{O}_x(1/N^2)$ and g is continuous. As $k \to \infty$, we certainly have $N = n^k \to \infty$, so (4) yields

$$\frac{g(x)}{x-1} = \frac{g(1)}{\ln x}, \quad \text{hence} \quad f(x) = \frac{f(1)}{x} \cdot \frac{x-1}{\ln x} \quad \text{for } x > 1, \tag{5}$$

since $g(1) = f(1)$. The only continuous function f on $[1, \infty)$ satisfying (5) is that given in (1) with $C = f(1)$, since $(x-1)/\ln x \to 1$ as $x \to 1$.

Conversely, let f be of the form (1), hence continuous on $[1, \infty)$. Equation (3) evidently holds for $x = 1$, and also for $x > 1$ since

$$nx^{n-1} f(x^n) = \frac{C}{x} \cdot \frac{x^n - 1}{\ln x} = \frac{C}{x \ln x}(x-1)(1 + x + \cdots + x^{n-1}) = f(x)(1 + x + \cdots + x^{n-1}).$$

Since equation (3) holds on $[1, \infty)$, both of the functions $x \mapsto \int_1^{x^n} f(t)dt$ and $x \mapsto \int_1^x (1 + t + \cdots + t^{n-1})dt$ have the same derivative on $[1, \infty)$; they also both have the value 0 at $x = 1$. Thus, f satisfies equation (2), hence also the functional equation in the statement of the problem.

Also solved by Robert Agnew, Hongwei Chen, Marty Getz & Yuanyuan Zhao, Tom Jager, Moubinool Omarjee (France), Paolo Perfetti (Italy), San Francisco University HS Problem Solving Group, Mehtaab Sawhney, Nicholas C. Singer, and the proposer. There were 3 incomplete or incorrect solutions.

An inequality between symmetric polynomials **June 2015**

1973. *Proposed by Arkady Alt, San Jose, California, USA.*

Let $\Delta(x, y, z) = 2(xy + yz + xz) - (x^2 + y^2 + z^2)$. Prove that for any positive real numbers a, b, and c, the following inequality holds:

$$\left(\Delta(a^2, b^2, c^2)\right)^2 \geq \Delta(a, b, c) \cdot \Delta(a^3, b^3, c^3).$$

Solution by Mehtaab Sawhney (student), Commack High School, Commack, NY.
We must prove $D(a, b, c) \geq 0$, where

$$D(a, b, c) = \left(\Delta(a^2, b^2, c^2)\right)^2 - \Delta(a, b, c) \cdot \Delta(a^3, b^3, c^3).$$

Since D is symmetric in the variables a, b, c, we may assume $a \leq b \leq c$ without loss of generality. Let $u = b - a$ and $v = c - a$; then we have $0 \leq u \leq v$. Let $P(a, u, v) = D(a, a + u, a + v)$, and let $\delta = v - u$. Direct computation shows that P is a polynomial in a, u, v of degree 6 in a, namely

$$P(a, u, v) = \sum_{j=0}^{6} Q_j(u, v)a^j.$$

The coefficients $Q_j = Q_j(u, v)$ are themselves polynomials in u, v, given by

$$Q_0 = \delta^4 uv\left(2u^2 + 3uv + 2v^2\right), \qquad Q_3 = 20\delta^2(u + v)(2u^2 + uv + 2v^2),$$
$$Q_1 = 2\delta^2(2u^5 + 7u^4 v + 7uv^4 + 2v^5), \qquad Q_4 = 10\delta^2(4u^2 + 5uv + 4v^2),$$
$$Q_2 = 20\delta^2(u^4 + 2u^3 v + 2uv^3 + v^4) + 19\delta^2 u^2 v^2, \qquad Q_5 = 2(u + v)(10\delta^2 + uv),$$
$$Q_6 = 4(\delta^2 + uv).$$

Since $u \geq v \geq 0$, we also have $\delta = v - u \geq 0$, hence $Q_j(u, v) \geq 0$, and so $D(a, b, c) = P(a, b - a, c - a) \geq 0$ for $0 \leq a \leq b \leq c$ as we sought to show.

Editor's Note. The inequality holds provided $a, b, c \geq 0$, as follows from the proof above, or else by continuity from the case $a, b, c > 0$.

Also solved by Andrea Fanchini (Italy), Kee-Wai Lau (Hong Kong), Paulo Perfetti (Italy), Nicholas Singer, and the proposer. There was 1 incomplete or incorrect solution.

A sum of reciprocals of *q*-polynomials **June 2015**

1974. *Proposed by Boon Wee Ong, Behrend College, Erie, PA.*

Let $q \neq 1$ be a positive real number. Define for $n \geq 1$,

$$\nu_n = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} \quad \text{and} \quad \mu_n = q^{n/2} + q^{-n/2}.$$

(a) Prove that $2\nu_{n+1} = \mu_1\nu_n + \mu_n$ for all $n \geq 1$,

(b) Prove that $2\nu_{n-1} = \mu_1\nu_n - \mu_n$ for all $n \geq 2$.

(c) Write the sum

$$\sum_{n=2}^{N} \frac{1}{\nu_n\nu_{n+1}}$$

in closed form.

Solution by M. Bello, M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal, Logroño, Spain. We have:

$$\begin{aligned}2\nu_{n+1} - \mu_n &= \frac{2(q^{(n+1)/2} - q^{-(n+1)/2}) - (q^{n/2} + q^{-n/2})(q^{1/2} - q^{-1/2})}{q^{1/2} - q^{-1/2}}\\ &= \frac{q^{(n+1)/2} - q^{-(n+1)/2} + q^{(n-1)/2} - q^{-(n-1)/2}}{q^{1/2} - q^{-1/2}}\\ &= \frac{q^{n/2}(q^{1/2} + q^{-/2}) - q^{-n/2}(q^{1/2} + q^{-1/2})}{q^{1/2} - q^{-1/2}} = \mu_1\nu_n,\end{aligned}$$

proving (a). For (b) we have

$$\begin{aligned}2\nu_{n-1} + \mu_n &= \frac{2(q^{(n-1)/2} - q^{-(n-1)/2}) + (q^{n/2} + q^{-n/2})(q^{1/2} - q^{-1/2})}{q^{1/2} - q^{-1/2}}\\ &= \frac{q^{(n-1)/2} - q^{-(n-1)/2} + q^{(n+1)/2} - q^{-(n+1)/2}}{q^{1/2} - q^{-1/2}} = \mu_1\nu_n,\end{aligned}$$

with the last equality justified by our proof of (a) above. For (c), start with the identity

$$\nu_{n+1}\mu_{n-1} - \mu_{n+1}\nu_{n-1} = \frac{2(q - q^{-1})}{q^{1/2} - q^{-1/2}} = 2\mu_1.$$

Using (a), this gives

$$\begin{aligned}\nu_{n+1}\nu_n - \nu_{n-1}\nu_{n+2} &= \nu_{n+1}\frac{\mu_1\nu_{n-1} + \mu_{n-1}}{2} - \nu_{n-1}\frac{\mu_1\nu_{n+1} + \mu_{n+1}}{2}\\ &= \frac{\nu_{n+1}\mu_{n-1} - \nu_{n-1}\mu_{n+1}}{2} = \mu_1.\end{aligned}$$

Therefore,

$$\frac{\mu_1}{\nu_{n+1}\nu_{n+2}} = \frac{\nu_{n+1}\nu_n - \nu_{n-1}\nu_{n+2}}{\nu_{n+1}\nu_{n+2}} = \frac{\nu_n}{\nu_{n+2}} - \frac{\nu_{n-1}}{\nu_{n+1}}$$

for $n \geq 2$. Letting $\nu_0 = 0$, the last identity also holds for $n = 1$. Hence,

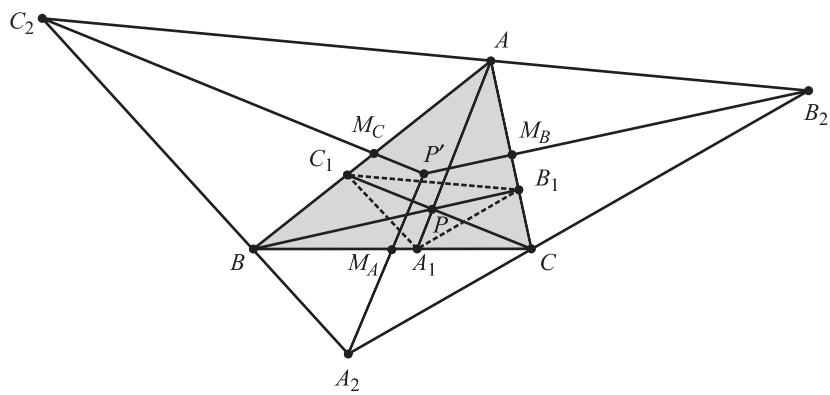
$$\sum_{n=2}^{N} \frac{1}{\nu_n\nu_{n+1}} = \frac{1}{\mu_1}\sum_{n=2}^{N}\left(\frac{\nu_{n-1}}{\nu_{n+1}} - \frac{\nu_{n-2}}{\nu_n}\right) = \frac{1}{\mu_1}\left(\frac{\nu_{N-1}}{\nu_{N+1}} - \frac{\nu_0}{\nu_2}\right) = \frac{\nu_{N-1}}{\mu_1\nu_{N+1}}.$$

Also solved by Adnan Ali (India), Michael Arakelian (Armenia), Michael Bacon & Charles K. Cook, Dionne Bailey & Elsie Campbell & Charles Diminnie, Michel Bataille (France), John Christopher, Michael Goldenberg & Mark Kaplan, Raymond N. Greenwell, Eugene A. Herman, Graham Lord, Dillon Montag, Northwestern University Math Problem Solving Group, Moubinool Omarjee (France), Mehtaab Sawhney, Joel Schlosberg, Jan Verster (Canada), Michael Vowe (Switzerland), John Zacharias, and the proposer. There were 2 incomplete or incorrect solutions.

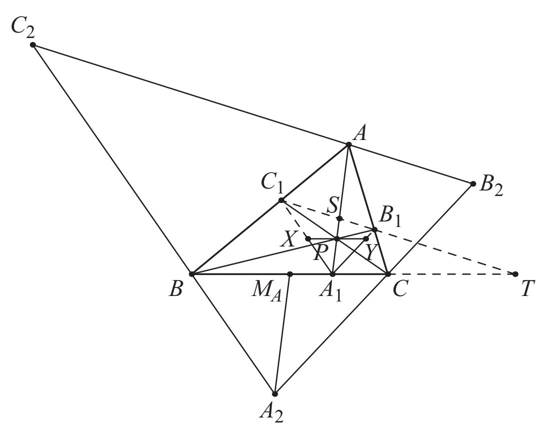
A homothety constructed via a cevian nest **June 2015**

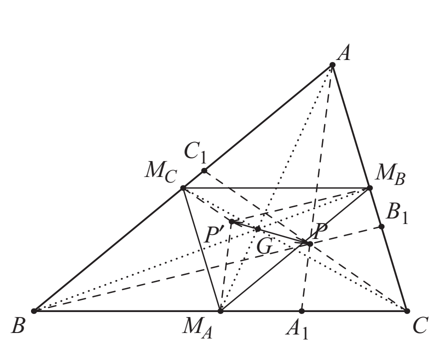
1975. *Proposed by Sohail Farhangi, Virginia Polytechnic Institute and State University, Blacksburg, VA.*

Let ABC be a triangle in the plane. For every point P in the interior of $\triangle ABC$, construct point P' as follows. Let lines AP, BP, and CP intersect sides BC, CA, and AB at points A_1, B_1, and C_1, respectively. Let L_A be the line passing through A that is parallel to B_1C_1, and define lines L_B and L_C in a similar manner. Let A_2 be the intersection of L_B and L_C, and define points B_2 and C_2 in a similar manner. Let M_A, M_B, and M_C be the midpoints of sides BC, CA, and AB, respectively. Finally, let P' be the concurrence point of the lines A_2M_A, B_2M_B, and C_2M_C (which are guaranteed to concur by the cevian nest theorem.) Prove that for any two points R and S inside $\triangle ABC$ the lines RS and $R'S'$ are parallel.



Solution by Dain Kim, Institute of Science Education for the Gifted and Talented, Seoul, Korea.





For the first part of this argument, refer to the left figure above. The line through P that is parallel to BC intersects A_1C_1 on X and A_1B_1 on Y. Let S, T be the intersection points of B_1C_1 with AA_1 and B_1C_1 with BC, respectively. It is well known that (B, A_1, C, T) and (C_1, S, B_1, T) are harmonic ranges. By perspective from A_1, (X, P, Y, ∞) is also harmonic, where ∞ is the point at infinity on the line XY; equivalently, P is the midpoint of $\overline{XY}$. Since $\triangle A_2BC$ and $\triangle A_1XY$ have corresponding sides parallel, they are homothetic, hence the homologous segments $\overline{A_1P}$ and $\overline{A_2M_A}$ are parallel. Thus, we have $AP \parallel M_AP'$, and similarly $BP \parallel M_BP'$ and $CP \parallel M_CP'$.

For the rest of the proof, refer to the right figure above. Let G be the centroid of $\triangle ABC$. The triangles $\triangle ABC$ and $\triangle M_AM_BM_C$ are homothetic from G with ratio $-2{:}1$. Since $AP \parallel M_AP'$, $BP \parallel M_BP'$ and $CP \parallel M_CP'$, it is clear that P and P' are homologous points under the homothety, so $\overrightarrow{GP} = -2\overrightarrow{GP'}$. Consequently, for any two points R, S and respective homologous points R', S', we have $\overrightarrow{RS} = -2\overrightarrow{R'S'}$. In particular, the lines RS and $R'S'$ are parallel.

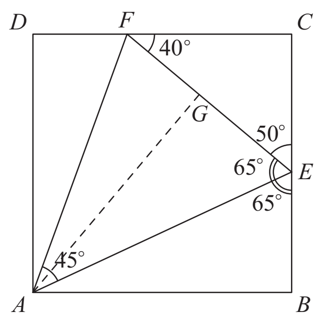
Editor's Note. The Cevian nest theorem was printed as Problem 1581 in this MAGAZINE (Vol. 72, No. 4, October 1999).

Also solved by Michel Bataille (France), Michael Goldenberg & Mark Kaplan, San Francisco University High School Problem Solving Team, Andrea Fanchini (Italy) and the proposer.

Answers

Solutions to the Quickies from page 294.

A1063. Label points as shown in the figure below. Fold the square along the line AE, placing $\triangle AEB$ atop $\triangle AEG$ where G is the reflection of B on AE. Since $AD = AB = AG$ and $\angle EAF = 45^\circ = \frac{1}{2}\angle BAD$, folding along AF brings $\triangle AFD$ atop $\triangle AFG$ (G is also the reflection of D on AF). Moreover, since $\angle ABE$ and $\angle ADF$ add up to a straight angle, G must lie on $\overline{EF}$. It follows that $\angle GEA = \angle AEB = 65^\circ$, $\angle FEC = 50^\circ$, and hence $\angle EFC = 40^\circ$.



Editor's Note. Given the measure $\alpha = \angle AEB$, the argument above may be used to show that $\angle CFE = 2\alpha - 90^\circ$.

A1064. Let $f(x) = x^{n-1} + \cdots + x + 1$. Note that $f(x) > 0$ for $x \in [0, 1]$. By the Cauchy–Schwartz inequality:

$$1 = \left(\int_0^1 \frac{\sqrt{f(x)}}{\sqrt{f(x)}}\,dx\right)^2 \le \int_0^1 f(x)\,dx \cdot \int_0^1 \frac{dx}{f(x)}.$$

Since $\int_0^1 f(x)\,dx = H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$, the claim $\int_0^1 dx/f(x) \ge 1/H_n$ follows.

REVIEWS

PAUL J. CAMPBELL, *Editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles, books, and other materials are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

Bolker, Ethan D., and Maura B. Mast, *Common Sense Mathematics*, MAA, 2016; xx+308 pp, $60 ($45 to MAA members). ISBN Print: 978-1-93951-210-9; eBook: 978-1-61444-621-7.

Jordan Ellenberg suggests in *How Not to Be Wrong* that "mathematics is the extension of common sense by other means." This new textbook for a QR (quantitative reasoning) course is a concrete manifestation of how and why that is so. Its examples and exercises are derived from "news of the day" involving numerical concepts; common sense is applied, with needed mathematics introduced on a just-in-time basis. Topics include back-of-the-envelope estimation, conversion of units, percentages, inflation, income distribution, reading a credit card bill, modeling climate change, exponential growth, lotteries, coincidences, and disease screening. A key consideration is that all of the examples and exercises deal with reality: real events with actual data bearing on genuine problems for people (and all sources are documented). Each chapter opens with a list of goals, and the exercises are classified by goal and by nature (routine, "worthy," complex,...). Hints are given for a few exercises, with the answers and solutions reserved to instructors. Halfway through, spreadsheets are introduced and the student is expected to create simple ones (caution: The instructions are keyed to a particular Windows version of Excel.) Most of what little algebraic notation that occurs is in connection with spreadsheets. The question arises: Shouldn't such "common sense" mathematics be taught to every high school student?

Wright, Tom, *Trolling Euclid: An Irreverent Guide to Nine of Mathematics' Most Important Problems*, CreateSpace Independent Publishing Platform (Amazon), 2016; iv+199 pp, $10.12(P) (print), $5.99 (Kindle). ISBN: 978-1-52346646-7

Definitely irreverent. Definitely worth it (but not necessarily for the irreverence). Wright's goal is to tell what mathematicians do, and his chosen vehicle is number theory—not just a little about primes and modular arithmetic, but the Big Stuff: the generalized Riemann hypothesis, the zeta function, the ABC conjecture, the Birch–Swinnerton-Dyer conjecture, and arithmetic progressions, with briefer discussions of the Collatz conjecture, the Goldbach conjecture, twin primes, and perfect numbers. He strips away all of the off-putting details of technicalities, leaving pure insight into the motivations and the results. Surprisingly, the reader needs to be able to deal with only an occasional equation, plus a more-than-occasional irreverent/irrelevant remark for the sake of entertainment.

Special Issue: Mathematical Depth. *Philosophia Mathematica* 23 (2) (June 2015).

This issue contains five essays plus a foreword and an afterword. Andrew Arana writes on the depth of Szemerédi's theorem, and the other authors plumb more generally what depth means in mathematics. They support the ideas that depth "ties together apparently disparate fields," "involves impurity [in definitions]," "finds order in chaos," "exhibits organizational or explanatory power," and/or "transforms a field or opens a new one."

Math. Mag. **89** (2016) 301–302. doi:10.4169/math.mag.89.4.301. © Mathematical Association of America

Ono, Ken, and Amir D. Aczel, *My Search for Ramanujan: How I Learned to Count*, Springer, 2016; xvi+238 pp, \$29.99 (print), \$5.99 (Kindle). ISBN Print: 978-3-319-25566-8; eBook: 978-3-319-25568-2.

Ono, famous for his resolution of a conjecture of Ramanujan about partitions, bares a detailed and excruciatingly intimate psychological portrait of his life, failures, and successes. Ono had Japanese immigrant parents who were "tiger" parents in their insistence on his achieving excellence, and much of his tale is about how he suffered and how he liberated himself from the resulting psychological handicap (which liberation reveals the double meaning of the subtitle). In a sense, he was "saved" by Ramanujan, whose life and achievements Ono describes; in fact, Ono was the mathematical consultant for the 2016 film about Ramanujan, *The Man Who Knew Infinity*. This is a gripping account of how Ono became the person he is.

Joseph, George Gheverghese, *Indian Mathematics: Engaging with the World from Ancient to Modern Times*, World Scientific, 2016; xvii+477 pp, \$98, \$48(P); ISBN 978-1-78634-060-3, 978-1-78634-061-0(P).

Joseph, author of *The Crest of the Peacock: Non-European Roots of Mathematics* (3rd ed., 2010), here elaborates on the history of mathematics in India, with special emphasis on the Kerala School and its impact (Joseph was born in Kerala). Keralese mathematicians devised results about infinite series and calculus before Europeans did; Joseph speculates about possible transmission of their work to Europe, although he admits the absence of "direct evidence." He emphasizes "circulatory histories over linear and bounded national or civilizational histories," as well as "transcendental sources of imagination, inspiration, commitment and resolve." He does not go into detail about this stance, except to note that "Knowledge is not something to be discovered; it circulates through revelations. . . . Discoveries are therefore particular moments in a history of knowledge in circulation."

Bliss, Karen M., Kathleen R. Fowler, and Benjamin J. Galluzzo, *Math Modeling: Getting Started and Getting Solutions* and *Math Modeling: Getting Started and Getting Solutions: Connections to the Common Core State Standards*, SIAM, 2014; 68 pp, 15 pp. Available from bookstore.siam.org, \$15. Free download at http://m3challenge.siam.org/about/mm/.

The first of these very colorful two booklets begins with an illustration of a difference between a "word problem" and a mathematical modeling problem: the latter has incomplete information and requires creativity. The booklet breaks modeling down into steps: defining the problem, making assumptions, defining variables, getting a solution, analysis and assessment, and reporting the results, with a short chapter on each and illustrative examples. The last 20 pages reproduce the problem and a student solution from the 2013 Moody's Mega Math Challenge contest, about the recycling of plastic bottles. The second booklet identifies the areas of mathematics, English, and science in the Common Core State Standards that are addressed by each of the steps in modeling. These booklets should be in the hands of every high school mathematics teacher, not just those who are coaching teams of contestants in modeling competitions.

Sundström, Manya Raman, Seduced by the beauty of mathematics, *New Scientist* (31 January 2015) 26–27.

Special Issue on the Nature and Experience of Mathematical Beauty, *Journal of Humanistic Mathematics* 6 (1) (January 2016), http://scholarship.claremont.edu/jhm/vol6/iss1/.

Raman-Sundström writes, "Do we have an inbuilt maths drive, a bit like a sex drive. . . ?. . . Can intellectual pursuit be as compelling as bodily urges?. . . [M]athematics might be just as satisfying as sex." Is the beauty of mathematics objective? At least we can agree that "teaching [mathematics] to students without conveying its beauty might be to miss the essence, the very life, of the subject." The questions led to the special issue indicated.

Correction: The discussion in the February 2016 column of László Babai's new algorithm for graph isomorphism contained two errors, which we set straight here: Graph isomorphism has never been proved NP-hard, and the running time for the best previous algorithm should have been $\exp\left(\mathcal{O}\sqrt{n \log n}\right)$, where n is the number of vertices. (Thanks to Eric Bach.) Also, the title of the *Comm. ACM* article cited should be "Algebraic fingerprints. . . ." (Thanks to Stan Wagon.)

45th United States of America Mathematical Olympiad and 7th United States of America Junior Mathematical Olympiad

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The United States of America Mathematics Olympiad (USAMO) and Junior Olympiad (USAJMO) are high-level contests in the style of the International Mathematical Olympiad (IMO) offered by the Committee on American Mathematics Competitions of the Mathematical Association of America. The two competitions are administered simultaneously, this year on April 19 and 20. Each competition uses the IMO format, consisting of three problems for each of two days, with an allowed time of 4.5 hours each day.

The USAMO is used to select a team of six students to represent the nation in the IMO, and the level of the problems reflects the level expected on the IMO competition. The USAJMO, offered to students in grade 10 and below, is used to identify students to train for participation in future IMO competitions. In setting problems for the USAJMO the Committee strives to provide a nicely balanced link between the computational character of the AIME problems and the more advanced proof-oriented problems of the USAMO.

The 2016 contests included two common problems. On the first day problem USAJMO3 was the same as USAMO1, and on the second day USAJMO6 and USAMO4 were identical.

This year 315 students sat for the USAMO contest, and 203 for the USAJMO. More information is available on the AMC section of the MAA website.

USAMO Problems

1. Let $X_1, X_2, \dots, X_{100}$ be a sequence of mutually distinct nonempty subsets of a set S. Any two sets X_i and X_{i+1} are disjoint and their union is not the whole set S, that is, $X_i \cap X_{i+1} = \emptyset$ and $X_i \cup X_{i+1} \neq S$, where $i \in \{1, \dots, 99\}$. Find the smallest possible number of elements in S.
2. Prove that for any positive integer k,
$$(k^2)! \cdot \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$
is an integer.

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3. Let $\triangle ABC$ be an acute triangle, and let I_B, I_C, and O denote its B-excenter, C-excenter, and circumcenter, respectively. Points E and Y are selected on $\overline{AC}$ such that $\angle ABY = \angle CBY$ and $\overline{BE} \perp \overline{AC}$. Similarly, points F and Z are selected on $\overline{AB}$ such that $\angle ACZ = \angle BCZ$ and $\overline{CF} \perp \overline{AB}$.

   Lines $\overline{I_B F}$ and $\overline{I_C E}$ meet at P. Prove that $\overline{PO}$ and $\overline{YZ}$ are perpendicular.
4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,

$$(f(x) + xy) \cdot f(x - 3y) + (f(y) + xy) \cdot f(3x - y) = (f(x + y))^2 .$$

5. An equilateral pentagon $AMNPQ$ is inscribed in triangle ABC such that $M \in \overline{AB}$, $Q \in \overline{AC}$, and $N, P \in \overline{BC}$. Let S be the intersection of $\overline{MN}$ and $\overline{PQ}$. Denote by ℓ the angle-bisector of $\angle MSQ$.

   Prove that $\overline{OI} \parallel \ell$, where O is the circumcenter of triangle ABC, and I is the incenter of triangle ABC.
6. Given are integers n and k, with $n \geq k \geq 2$. You play the following game against an evil wizard.

   The wizard has $2n$ cards; for each $i = 1, \ldots, n$, there are two cards labeled i. Initially, the wizard places all cards face down in a row, in unknown order.

   You may repeatedly make moves of the following form: you point to any k of the cards. The wizard then turns those cards face up. If any two of the cards match, the game is over and you win. Otherwise, you must look away, while the wizard arbitrarily permutes the k chosen cards and then turns them back face down. Then, it is your turn again.

   We say this game is *winnable* if, for some m, you have a strategy that guarantees you will win in at most m moves.

   For which values of n and k is the game winnable?

Solutions

1. Since we must have $2^{|S|} \geq 100$, we must have $|S| \geq 7$. We will provide an inductive construction for a *chain* of subsets $X_1, X_2, \ldots, X_{2^{n-1}+1}$ of $S = \{1, \ldots, n\}$ satisfying $X_i \cap X_{i+1} = \emptyset$ and $X_i \cup X_{i+1} \neq S$ for each $n \geq 4$.

   For $S = \{1, 2, 3, 4\}$, the following chain of length $2^3 + 1 = 9$ will work:

$$\{3, 4\} \quad \{1\} \quad \{2, 3\} \quad \{4\} \quad \{1, 2\} \quad \{3\} \quad \{1, 4\} \quad \{2\} \quad \{1, 3\}.$$

   Now, given a chain of subsets of $\{1, 2, \ldots, n\}$ the following procedure produces a chain of subsets of $\{1, 2, \ldots, n + 1\}$:

   (a) take the original chain, delete any element, and make two copies of this chain, which now has even length;

   (b) glue the two copies together, joined by $\varnothing$ in between; and then

   (c) insert the element $n + 1$ into the sets in alternating positions of the chain starting with the first.

   It can be easily checked that if the original chain satisfies the requirements, then so does the new chain, and if the original chain has length $2^{n-1} + 1$, then the new chain has length $2^n + 1$, as desired. This construction yields a chain of length 129 when $S = \{1, 2, \ldots, 8\}$.

   To see that $|S| = 8$ is the minimum possible size, consider a chain on the set $S = \{1, 2, \ldots, 7\}$ satisfying $X_i \cap X_{i+1} = \emptyset$ and $X_i \cup X_{i+1} \neq S$. Because of these requirements any subset of size 4 or more can only be neighbored by sets of size 2 or less, of which there are $\binom{7}{1} + \binom{7}{2} = 28$ available. Thus, the chain can contain no more than 29 sets of size 4 or more and no more than 28 sets of size 2 or less. Finally, since there are only $\binom{7}{3} = 35$ sets of size 3 available, the total number of sets in such a chain can be at most $29 + 28 + 35 = 92 < 100$.

This problem was proposed by Iurie Boreico.

2. We show the exponent of any given prime p is nonnegative in the expression. Recall that the exponent of p in $n!$ is equal to $\sum_{i\geq 1} \lfloor n/p^i \rfloor$. In light of this, it suffices to show that for any prime power P, we have

$$\left\lfloor \frac{k^2}{P} \right\rfloor \geq \sum_{j=0}^{k-1} \left(\left\lfloor \frac{j+k}{P} \right\rfloor - \left\lfloor \frac{j}{P} \right\rfloor \right).$$

Since both sides are integers, we show

$$\left\lfloor \frac{k^2}{P} \right\rfloor > -1 + \sum_{j=0}^{k-1} \left(\left\lfloor \frac{j+k}{P} \right\rfloor - \left\lfloor \frac{j}{P} \right\rfloor \right).$$

If we denote by $\{x\}$ the fractional part of x, then $\lfloor x \rfloor = x - \{x\}$, and so the previous inequality can be rewritten

$$\left\{ \frac{k^2}{P} \right\} + \sum_{j=0}^{k-1} \left\{ \frac{j}{P} \right\} < 1 + \sum_{j=0}^{k-1} \left\{ \frac{j+k}{P} \right\}.$$

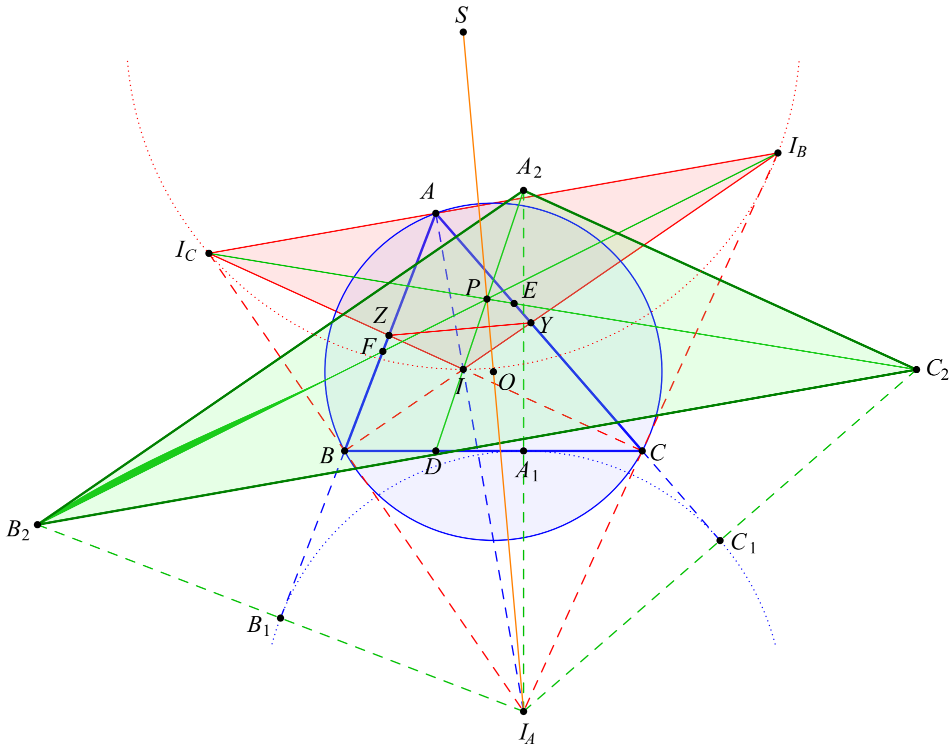
However, the sum of remainders when $0, 1, \ldots, k-1$ are taken modulo P is easily seen to be less than the sum of remainders when $k, k+1, \ldots, 2k-1$ are taken modulo P. So

$$\sum_{j=0}^{k-1} \left\{ \frac{j}{P} \right\} \leq \sum_{j=0}^{k-1} \left\{ \frac{j+k}{P} \right\}$$

follows, and we are done upon noting $\left\{ k^2/P \right\} < 1$.

This problem was proposed by Kiran Kedlaya.

3. Let I_A denote the A-excenter and I the incenter. Then let D denote the foot of the altitude from A. Suppose the A-excircle is tangent to $\overline{BC}$, $\overline{AB}$, $\overline{AC}$ at A_1, B_1, C_1 and let A_2, B_2, C_2 denote the reflections of I_A across these points. Let S denote the circumcenter of $\triangle I I_B I_C$.



We begin with the following observation: points D, I, A_2 are collinear, points E, I_C, C_2 are collinear, and points F, I_B, B_2 are collinear. This follows from the "midpoints of altitudes" lemma.

Observe that $\overline{B_2C_2} \parallel \overline{B_1C_1} \parallel \overline{I_BI_C}$. Similarly, on the other sides we discover $\triangle II_BI_C$ and $\triangle A_2B_2C_2$ are homothetic. Hence, P is the center of this homothety—in particular, D, I, P, A_2 are collinear. Moreover, P lies on the line joining I_A to S, which is the Euler line of $\triangle II_BI_C$, so it passes through the nine-point center of $\triangle II_BI_C$, which is O. Consequently, P, O, I_A are collinear as well.

To finish, we need only prove that $\overline{OS} \perp \overline{YZ}$. In fact, we claim that $\overline{YZ}$ is the radical axis of the circumcircles of $\triangle ABC$ and $\triangle II_BI_C$. Actually, Y is the radical center of these two circumcircles and the circle with diameter $\overline{II_B}$, which passes through A and C. Analogously, Z is the radical center of the circumcircles and the circle with diameter $\overline{II_C}$, and the proof is complete.

This problem was proposed by Evan Chen.

4. First, taking $x = y = 0$ in the given yields $f(0) = 0$, and then taking $x = 0$ gives $f(y)f(-y) = f(y)^2$. So also $f(-y)^2 = f(y)f(-y)$, from which we conclude f is even. Then taking $x = -y$ gives

$$\forall x \in \mathbb{R}: \quad f(x) = x^2 \quad \text{or} \quad f(4x) = 0 \quad (\star)$$

for all x.

Next, we claim that

$$\forall x \in \mathbb{R}: \quad f(x) = x^2 \quad \text{or} \quad f(x) = 0 \quad (\heartsuit).$$

To see this assume $f(t) \neq 0$ (hence $t \neq 0$). By $(\star)$ we get $f(t/4) = t^2/16$. Now take $(x, y) = (3t/4, t/4)$ to get

$$\frac{t^2}{4} f(2t) = f(t)^2 \implies f(2t) \neq 0.$$

If we apply ($\star$) again we actually also get $f(t/2) \neq 0$. Together these imply

$$f(t) \neq 0 \iff f(2t) \neq 0 \quad (\spadesuit).$$

Repeat ($\spadesuit$) to get $f(4t) \neq 0$, hence $f(t) = t^2$, proving ($\heartsuit$).

We are now ready to prove the claim that the only two functions satisfying the requirements are $f(x) = 0$ for all $x \in \mathbb{R}$ and $f(x) = x^2$ for all $x \in \mathbb{R}$.

Assume there's an $a \neq 0$ for which $f(a) = 0$; we show that $f \equiv 0$.

Let $b \in \mathbb{R}$ be given. Since f is even, we can assume without loss of generality that $a, b > 0$. Also, note that $f(x) \geq 0$ for all x by ($\heartsuit$). By using ($\spadesuit$) we can generate $c > b$ such that $f(c) = 0$ by taking $c = 2^n a$ for a large enough integer n. Now, select $x, y > 0$ such that $x - 3y = b$ and $x + y = c$. That is,

$$(x, y) = \left(\frac{3c + b}{4}, \frac{c - b}{4}\right).$$

Substitution into the original equation gives

$$0 = (f(x) + xy)\, f(b) + (f(y) + xy)\, f(3x - y) = (f(x) + f(y) + 2xy)\, f(b).$$

Since $f(x) + f(y) + 2xy > 0$, if follows that $f(b) = 0$, as desired.

This problem was proposed by Titu Andreescu.

5. Let δ and ε denote $\angle MNB$ and $\angle CPQ$. Also, assume $AMNPQ$ has side length 1.

In what follows, assume $AB < AC$. First, we note that

$$BN = (c - 1)\cos B + \cos \delta,$$

$$CP = (b - 1)\cos C + \cos \varepsilon, \text{ and}$$

$$a = 1 + BN + CP$$

from which it follows that

$$\cos \delta + \cos \varepsilon = \cos B + \cos C - 1.$$

Also, by the Law of Sines, we have $\frac{c-1}{\sin \delta} = \frac{1}{\sin B}$ and similarly on triangle CPQ, and from this we deduce

$$\sin \varepsilon - \sin \delta = \sin B - \sin C.$$

The sum-to-product formulas

$$\sin \varepsilon - \sin \delta = 2 \cos\left(\frac{\varepsilon + \delta}{2}\right) \sin\left(\frac{\varepsilon - \delta}{2}\right)$$

$$\cos \varepsilon + \cos \delta = 2 \cos\left(\frac{\varepsilon + \delta}{2}\right) \cos\left(\frac{\varepsilon - \delta}{2}\right)$$

give us

$$\tan\left(\frac{\varepsilon - \delta}{2}\right) = \frac{\sin \varepsilon - \sin \delta}{\cos \varepsilon + \cos \delta} = \frac{\sin B - \sin C}{\cos B + \cos C - 1}.$$

Now note that ℓ makes an angle of $\frac{1}{2}(\pi + \varepsilon - \delta)$ with line BC. Moreover, if line OI intersects line BC with angle φ then

$$\tan \varphi = \frac{r - R\cos A}{\frac{1}{2}(b - c)}.$$

So in order to prove the result, we only need to check that

$$\frac{r - R\cos A}{\frac{1}{2}(b-c)} = \frac{\cos B + \cos C - 1}{\sin B - \sin C}.$$

Using the fact that $b = 2R\sin B$, $c = 2R\sin C$, this reduces to the fact that $r/R + 1 = \cos A + \cos B + \cos C$, which is the so-called Carnot theorem.

This problem was proposed by Ivan Borsenco.

6. The game is winnable if and only if $k < n$.

   First, suppose $2 \le k < n$. Query the cards in positions $\{1, \ldots, k\}$, then $\{2, \ldots, k+1\}$, and so on, up to $\{2n-k+1, 2n\}$. By taking the difference of any two adjacent queries, we can deduce for certain the values on cards $1, 2, \ldots, 2n-k$. If $k \le n$, this is more than n cards, so we can find a matching pair.

   For $k = n$ we remark the following: at each turn after the first, assuming one has not won, there are n cards representing each of the n values exactly once, such that the player has no information about the order of those n cards. We claim that consequently the player cannot guarantee victory. Indeed, let S denote this set of n cards, and $\overline{S}$ the other n cards. The player will never win by picking only cards in S or $\overline{S}$. Also, if the player selects some cards in S and some cards in $\overline{S}$, then it is possible that the choice of cards in S is exactly the complement of those selected from $\overline{S}$; the strategy cannot prevent this since the player has no information on S. This implies the result.

   Note: In the problem statement, if the sentence, "The wizard then turns those cards face up," more specifically states, "The wizard then collects these cards, shuffles them randomly, and turns those cards face up," the solution to the problem is different but also interesting.

   This problem was proposed by Gabriel Carroll.

USAJMO Problems

1. The isosceles triangle $\triangle ABC$, with $AB = AC$, is inscribed in the circle ω. Let P be a variable point on minor arc $\overset{\frown}{BC}$, and let I_B and I_C denote the incenters of triangles $\triangle ABP$ and $\triangle ACP$, respectively.

   Prove that as P varies, the circumcircle of triangle $\triangle PI_BI_C$ passes through a fixed point.
2. Prove that there exists a positive integer $n < 10^6$ such that 5^n has six consecutive zeros in its decimal representation.
3. Same as USAMO 1.
4. Find, with proof, the least integer N such that if any 2016 elements are removed from the set $\{1, 2, \ldots, N\}$, one can still find 2016 of the remaining elements with sum N.
5. Let $\triangle ABC$ be an acute triangle, with O as its circumcenter. Point H is the foot of the perpendicular from A to line $\overleftrightarrow{BC}$, and points P and Q are the feet of the perpendiculars from H to the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$, respectively.

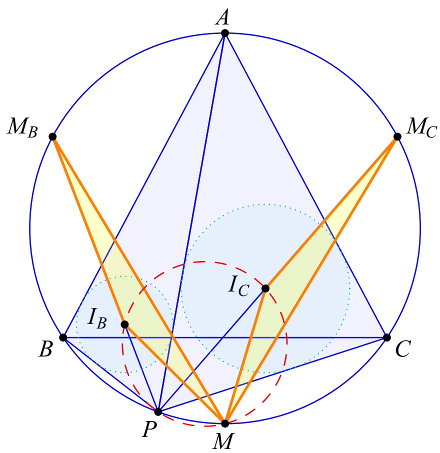
   Given that

   $$AH^2 = 2 \cdot AO^2,$$

   prove that the points O, P, and Q are collinear.
6. Same as USAMO 4.

Solutions

1. Let M be the midpoint of arc BC not containing A. We claim M is the desired fixed point.



Since $\angle MPA = 90^\circ$ and ray PA bisects $\angle I_B P I_C$, it suffices to show that $MI_B = MI_C$. Let M_B, M_C be the second intersections of PI_B and PI_C with circumcircle. Now $M_B I_B = M_B B = M_C C = M_C I_C$, and moreover $MM_B = MM_C$, and $\angle I_B M_B M = \frac{1}{2}\widehat{PM} = \angle I_C M_C M$, so $\triangle I_B M_B M \cong \triangle I_C M_C M$.

This problem was proposed by Ivan Borsenco and Zuming Feng.

2. One answer is $n = 20 + 2^{19} = 524308$.

First, observe that

$$5^n \equiv 5^{20} \pmod{5^{20}}$$
$$5^n \equiv 5^{20} \pmod{2^{20}}$$

the former being immediate and the latter since $\varphi(2^{20}) = 2^{19}$. Hence $5^n \equiv 5^{20} \pmod{10^{20}}$. Moreover, we have

$$5^{20} = \frac{1}{2^{20}} \cdot 10^{20} < \frac{1}{1000^2} \cdot 10^{20} = 10^{-6} \cdot 10^{20}.$$

Thus the last 20 digits of 5^n will begin with six zeros.

This problem was proposed by Evan Chen.

3. Same as USAMO 1.

4. The answer is

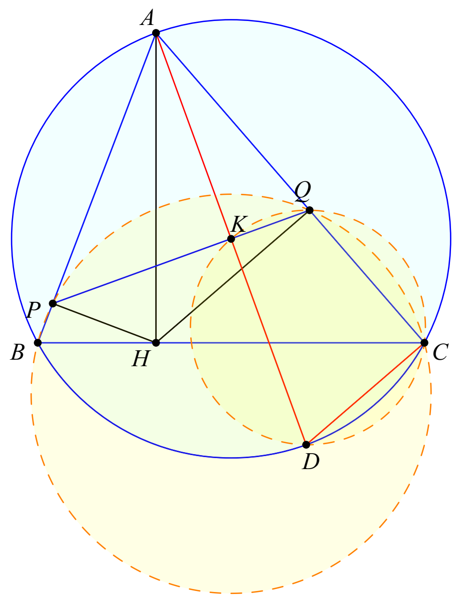
$$N = 2017 + 2018 + \cdots + 4032 = 1008 \cdot 6049 = 6097392.$$

To see that N must be at least this large, consider the situation when $1, 2, \ldots, 2016$ are removed. Among the remaining elements, any sum of 2016 elements is certainly at least $2017 + 2018 + \cdots + 4032$.

Now we show this value of N works. Consider the 3024 pairs of numbers $(1, 6048)$, $(2, 6047)$, $\ldots$, $(3024, 3025)$. Regardless of which 2016 elements of $\{1, 2, \ldots, N\}$ are deleted, at least $3024 - 2016 = 1008$ of these pairs have both elements remaining. Since each pair has sum 6049, we can take these pairs to be the desired numbers.

This problem was proposed by Gregory Galperin.

5. First, since $AP \cdot AB = AH^2 = AQ \cdot AC$, it follows that $PQCB$ is cyclic. Consequently, we have $AO \perp PQ$.



Let K be the foot of A onto PQ, and let D be the point diametrically opposite A. Thus A, K, O, D are collinear.

Since quadrilateral $KQCD$ is cyclic ($\angle QKD = \angle QCD = 90°$), we have

$$AK \cdot AD = AQ \cdot AC = AH^2 \implies AK = \frac{AH^2}{AD} = \frac{AH^2}{2AO} = AO$$

so $K = O$.

This problem was proposed by Jacek Fabrykowski and Zuming Feng.

6. Same as USAMO 4.

The top fourteen students on the 2016 USAMO were (in alphabetical order):

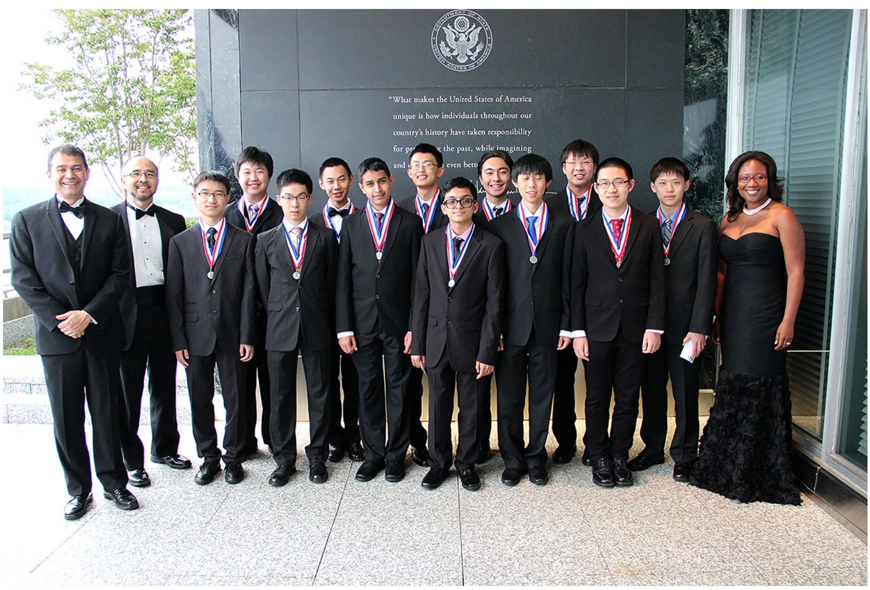
Ankan Bhattacharya	11	Lawrence Technological University	MI
Ruidi Cao	11	Missouri Academy of Science, Math, and Computing	MO
Hongyi Chen	10	University of Colorado at Boulder	CO
Jacob Klegar	12	Choate-Rosemary Hall School	CT
James Lin	11	Winchester High School	MA
Allen Liu	12	Penfield Senior High School	NY
Junyao Peng	11	Princeton International School of Math and Science	NJ
Kevin Ren	10	Torrey Pines High School	CA
Mihir Singhal	10	Palo Alto Senior High School	CA
Alec Sun	11	Phillips Exeter Academy	NH
Kevin Sun	12	Phillips Exeter Academy	NH
Alexander Wei	12	Phillips Exeter Academy	NH
Allen Yang	11	Lakeside High School	WA
Yuan Yao	11	Phillips Exeter Academy	NH

The top thirteen students on the 2016 USAJMO were (in alphabetical order):

Evan Fang	10	Lexington High School	MA
Eric Gan	8	A & M Consolidated High School	TX
Wanlin Li	10	Syosset High School	NY
Andrew Lin	10	Lynbrook High School	CA
Jason Liu	8	Davidson Academy of Nevada	NV
Kevin Liu	9	Carmel Senior High School	IN
Kevin Qian	9	Montgomery Blair High School	MD
Nathan Ramesh	9	Lexington High School	MA
Carl Schildkraut	9	Lakeside High School	WA
Colin Tang	9	Interlake High School	WA
Edward Wan	7	Lakeside High School	WA
Brandon Wang	8	Redwood Middle School	CA
Catherine Wu	9	Saratoga High School	CA

IN MEMORIAM: Jacek Fabrykowski, Professor of Mathematics, Youngstown State University, was chair of the USAMO Committee from 2010 until his death on July 12, 2016. He first joined the American Mathematics Competition problems group when he was appointed in 2003 as a member of the American Invitational Mathematics Examination Committee, on which he served for 13 years. In appreciation of his exemplary service to the mathematics community we dedicate this year's USAMO and USAJMO competitions to the memory of Jacek Fabrykowski.

United States of America Mathematical Olympiad



USAMO winners and speakers. From L to R: Dr. Jesús De Loera (evening speaker), Dr. Francis Su (MAA President), Yuan Yao, James Lin, Allen Liu, Ruidi Cao, Mihir Singhal, Kevin Ren, Ankan Bhattacharya, Jacob Klegar, Kevin Sun, Alec Sun, Junyao Peng, Hongyi Chen, Dr. Talithia Williams (morning speaker).

Carl B. Allendoerfer Awards

The Carl B. Allendoerfer Awards, established in 1976, are made to authors of articles of expository excellence published in *Mathematics Magazine*. The Awards are named for Carl B. Allendoerfer, a distinguished mathematician at the University of Washington and President of the Mathematical Association of America, 1959–60.

Julia Barnes, Clinton Curry, Elizabeth Russell, and Lisbeth Schaubroeck

"Emerging Julia Sets," *Mathematics Magazine*, Volume 88, Number 2, April 2015, pages 91-102.

The study of discrete dynamics, taking iterations of a simple function from the complex plane to itself, is a wonderful example of rich complexity arising from the simplest of contexts. This well-written and visually intriguing paper provides a novel approach to the mathematics of Julia sets. The authors decompose a function $f\colon \mathbb{C} \to \mathbb{C}$ and its iterates into real and imaginary parts and then explore the connections between the graphs of the real and imaginary parts and their corresponding filled Julia sets. This approach conveys a significant amount of information about the iterates.

The article begins with a summary of results about any nonconstant, complex analytic function, the graph of which cannot have maximum or minimum values so the surface is unbounded above and below. All critical points are saddle points, and the surface consists of waves between the critical points. When the original function is iterated, more saddle points appear; this increases the number of undulations on the graphs. The authors capture and visually illustrate this behavior.

The first examples the authors consider are simple quadratics; they discuss the filled Julia set and compare it to its corresponding real and imaginary analogs. The authors extend their results to rational functions, where the images are much more intricate and carry names like checkerboard, perturbed rat, and Cantor sets of circles. The article includes a nice illustration of the Riemann mapping theorem using images of Julia sets. Throughout the article, intricate visualizations illustrate the theory developed.

The results presented in this paper are interesting and the visualizations are impressive. The clever choice of examples leads the reader through the complicated behavior of iterated complex-valued functions.

Response from Barnes, Curry, Russell, and Schaubroeck

We are humbled and honored to win the Carl B. Allendoerfer Award for our paper, "Emerging Julia Sets." This project began when Beth and Julie, longtime friends, explored connections between Beth's complex analysis and Julie's complex dynamics, producing intriguing images. Julie presented initial findings at an MAA-SE Section meeting; Clinton was there and joined the project. Later, Julie and Elizabeth met at a teaching workshop in New York, escaped to an outdoor art museum to discuss math,

Math. Mag. **89** (2016) 312–315. doi:10.4169/math.mag.89.4.312. © Mathematical Association of America

and called Beth in Colorado, asking her to plot more graphs! The four of us, from different time zones and different career stages, continued collaborations during various MAA meetings, including a PREP workshop at West Point and JMM in San Diego. Clearly the opportunities provided by the MAA for us to interact cannot be overstated! We are grateful to the editor and referees of *Mathematics Magazine,* whose suggestions improved our exposition. Finally, we thank the Allendoerfer Award Committee for recognizing our work with this honor.

Biographical Notes

Julia Barnes earned her Ph.D. from the University of North Carolina at Chapel Hill under the direction of Jane Hawkins. Her academic training is a cross between complex dynamics and ergodic theory; she has published in both areas as well as on the mathematics of weather and on using hands-on teaching ideas. Julie is a Professor of Mathematics at Western Carolina University and has taught there since 1996, except for one year as a Distinguished Visiting Professor at the United States Air Force Academy. She is currently one of the Associate Directors for MAA Project NExT, serves as chair of the MAA Committee on Professional Development, and is on the PRIMUS Board of Directors. More locally, she creates noncompetitive problem solving opportunities for undergraduates by organizing a mathematical treasure hunt for her section meeting and serving as Math Club advisor at her university. In her spare time, she enjoys hiking and playing racquetball.

Clinton Curry received his Ph.D. in 2009, having studied topology and dynamical systems under John C. Mayer and Alexander M. Blokh at the University of Alabama at Birmingham. Clinton was a visiting lecturer at Stony Brook University and an assistant professor at Huntingdon College before he decided to turn his computer programming hobby into a computer programming career. Clinton is now a software engineer at Google, where he misses having his own office, but still learns something every single day. In his spare time, Clinton enjoys playing guitar and exploring the San Francisco Bay Area with his wife Jessica.

Elizabeth Russell has been fascinated by the subject of complex dynamics ever since her undergraduate days at Hofstra University. Her love of chaos continued through a Ph.D. at Boston University where she studied under the direction of Bob Devaney. Since graduation, Liz has held a number of positions including work with the federal government and academic positions at both the United States Military Academy and Western New England University. Liz spends her spare time practicing yoga and burning the floor as a competitive ballroom dancer. She lives in Maryland with her husband Joshua.

Lisbeth Schaubroeck earned her Ph.D. from the University of North Carolina at Chapel Hill in 1998 under the direction of the late John Pfaltzgraff. Although her academic training is in geometric function theory, she has published in a wide range of fields, including polynomial roots and the mathematics of weather. Beth's academic career has been at the United States Air Force Academy in Colorado Springs, where she has worked as faculty development director for her department, co-coordinator for the mathematics major, and advisor for academic strategy. She enjoys teaching courses across the curriculum, from freshman calculus to senior complex analysis, and has mentored several student projects in elementary knot theory. She and Tim, her husband of 26 years, have two sons who keep their lives busy—most recently with a lot of target archery. Beth is trying to master the skill, but the little tiny bullseye still eludes her.

Irl Bivens and Ben Klein

"The Median Value of a Continuous Function," *Mathematics Magazine*, Volume 88, Number 1, February 2015, pages 39-51.

In "The Median Value of a Continuous Function," Bivens and Klein take a familiar topic from introductory statistics and apply it to continuous functions in an intuitive and beautiful way. Building off the common calculus topic of the mean value of a function, the authors develop the notion of the median value of a function and immediately show the connection between their definition and a natural area minimization problem. The concept and its utility spring to life in an enlightening and entertaining way as the authors reveal many surprising results. While acknowledging that the median concept is well known in measure theory, the authors define the concept in a straightforward manner that uses only basic ideas from calculus and undergraduate analysis. This approach brings the topic within reach of first year calculus students. The writing is engaging and highly accessible, so that the paper is suitable reading for undergraduates. The authors have created a rich set of supplementary examples and exercises available online, which provide a great resource for readers interested in learning more and for faculty looking for good undergraduate projects.

In this article, the choices of examples and accompanying figures are well designed to help in understanding the exposition and development. Examples include area minimization and a clever problem involving expected value and the infamous Wile E. Coyote. After exploring various aspects of the median in the context of area, Bivens and Klein lead the reader to make the connection to measure theory. They relate Lebesgue measure to easily defined sets and provide an alternative characteristic of the median of a function, and in the process introduce sophisticated mathematics in a nonintimidating setting.

Bivens and Klein have deftly woven a path across mathematical levels and areas. With so many deep connections to familiar topics, there is something to grab the interest of a wide variety of mathematicians and students.

Response from Irl Bivens and Ben Klein

We are deeply honored that our article has received a Carl B. Allendoerfer Award. Over the course of our careers we have worked together on many projects and this recognition is, without question, the high point of our collaborations. A special debt of gratitude is owed to Walter Stromquist, who shepherded our manuscript through the review and publication process. Walter epitomizes the best qualities of an editor, and coordinating with him was an absolute pleasure.

Our study of the median had its roots in a seemingly innocent calculus problem, to find the line through the origin that minimizes the area between that line and the graph of a given continuous function. We discovered that solutions to this problem exhibit a surprising commonality. Our attempts to understand this commonality spanned a multitude of years that witnessed the writing of more than a dozen manuscripts on the topic, the retirement of one of us (Ben), and the soon-to-be retirement of the other. We are grateful to our Department for its unending support over this time, and to the two Academic Deans who patiently tolerated the "in progress" annotation to our work on the median year after year. Finally, we express a sincere "thank you" to the Southeastern Section of the MAA for allowing us the (at least) four special session talks on the median we've given over the past decade.

Biographical Notes

Irl Bivens received his A.B. degree from Pfeiffer College and his Ph.D. from the University of North Carolina at Chapel Hill. After teaching at Pfeiffer College and Rice University, he joined the faculty of Davidson College in 1982. His work with the Mathematical Association includes a term as the Section Lecturer for the Southeastern Section, fifteen years on the Board of Editors for *The College Mathematics Journal*, and service on numerous committees and subcommittees associated with publications of the MAA. For a total of fourteen years, he and his colleague, Ben Klein, wrote the North Carolina State Mathematics Contest. At Davidson he has taught a variety of courses including differential geometry, courses on mathematics and magic, and a popular seminar on the history of mathematics. For relaxation he swims, reads, and attempts to juggle.

Ben Klein received his bachelor's degree in mathematics from the University of Rochester and then earned an M.A. and Ph.D. (under G. A. Hedlund) in mathematics from Yale University. After four years in the Department of Mathematics at New York University, he moved to North Carolina and served on the faculty of Davidson College for thirty-seven years. He retired from full-time teaching in 2008 but has taught, off and on, on a part-time basis ever since. His involvement with the Mathematical Association of America includes terms first as chair and then as governor of the Southeastern Section. For five years, in collaboration with his colleagues Irl Bivens and Richie King, Klein edited the Problems and Solutions Section of *The College Mathematics Journal*. He has also been active with the North Carolina Council of Teachers of Mathematics and the Advanced Placement Calculus Program.

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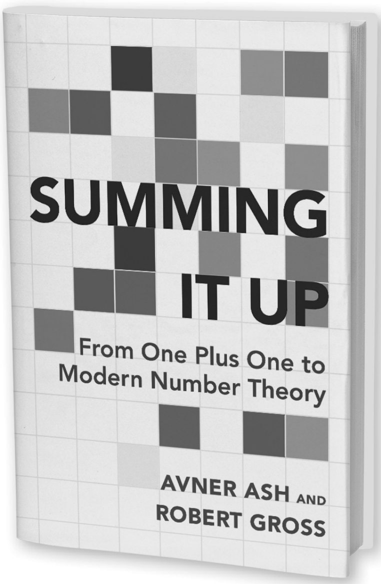

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